Classical and Quantum Cosmology

Daniel Panizo

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1 Classical Cosmology

Several satellites that we have sent out there,[1] into the empty and cold darkness of space, have provided enough data to prove what Hubble already stated in 1929 [1]: Every galaxy far away (and not so far away) from us is in the process of getting further away. What is more, their distance is increasing even faster today that it was yesterday [2, 3]. It could be, perhaps, due to the fact that the Universe itself sees human beings as a potential plague, and wants to avoid us. Or could be that some sort of colossal multidimensional being has decided to stretch the fabric of spacetime itself, just for fun. Nonsense aside, the most widely accepted reason for the accelerating expansion of the universe today is *Dark Energy*. And what is *Dark Energy*? Where does it come from? What it is and its origin are still open questions. But let us first go back more than a century ago, to understand the synthesis of our current comprenhesion of the cosmos.

Einstein published his theory of *General Relativity* (GR) [4, 5] in 1915. The world did not become a better place due to this, but at least we were provided with a ridiculous powerful tool to describe gravitational events. In years following its publication, Friedmann, Hubble, Lemaître [1, 6, 7] (among many others) used general relativity technology to describe the Universe as a whole. Their work set the foundations for what we call today *The Standard Model of Cosmology* (a.k.a. Λ-CDM model) [2,8,9].[2] There are three main components that give name to this model:

- Dark energy, an unknown form of energy related to the vacuum expectation value, that

[1] Plus all evidence collected also from the Earth's surface.
[2] Worthy of a Nobel prize, awarded to Jim Peebles in 2019.

causes spacetime to expand at an accelerated rate. The cosmological constant Λ, which Einstein introduced in his equations has been reassigned to represent the dark energy density.

- Cold Dark Matter, a hypothetical non-relativistic type of matter with no known interactions with regular matter other than gravitational. Its presence can be inferred by the measuring rotational velocities of spiral galaxies [10, 11].

- Regular relativistic and non-relativistic ordinary matter, such as electromagnetic radiation and the atoms that form observable structures like ourselves.

This model can provide a resonably good account of the following observed features of the universe:

- The existence of the Comic Microwave Background (CMB), remnant radiation from the early universe that was released as the early hot universe cooled down, creating a homogenous wall of radiation, "visible" at radio frequencies (see the CMB temperature distribution in figure 1).

- The observed abundances of light elements such as hydrogen, helium and lithium. Predictions of these distributions can be compared with the measured power spectrum and anisotropies of the CMB, which contain observational evidence for these abundances produced during the nucleosynthesis process.[3]

- Large-scale structures in the distribution of galaxies.

- The observed accelerating expansion of the universe at large scales.

This model is good enough to describe our current observations of the Universe. And not only that. If we reversed the observed expansion back to very close to the beginning of everything, we could still have a really good description of the events happening in the almost newborn Universe. This model has four really simple foundations:

1. *Copernican*: Our planet occupies no special position in the Universe.

2. *GR + Expansion*: The assumption that the gravitational dynamics of the universe are correctly described by Einstein equation.

3. *Hubble's discovery*: The observations performed during 1929 by Edwin Hubble showed that the greater the distance between any two galaxies, the greater their relative speed of separation [1]. As Lemaître later showed, it is not that galaxies are moving away from each other, but spacetime stretching, i.e. the universe is expanding [12].

4. *Perfect fluidity*: We can assume that all the contents of the Universe behave like a perfect fluid, i.e. they are not sticky.

Of course this model has its flaws, but we leave these downsides for future lines in section 1.2. At really big scales, Copernican principle holds. No point in space has a special position. Wherever you sit at and look at, everything is more or less the same. In technical words, this means *homogeneity* and *isotropy*. The next step is to find a reliable way to measure distances so that we can describe the geometry of spacetime. This is given by the *line invariant*, which takes the famous *FLRW*[4] form, adequate to describe a Lorentzian signature spacetime with a

[3] The production of light nuclei other than the hydrogen during the early phases of the universe.

[4] Friedmann-Lemaître-Robertson-Walker.

high degree of symmetry like the one we seem to live in. This can be written as:

$$\begin{aligned} \mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a\mathrm{d}x^b \\ &= -N^2(t)\mathrm{d}t^2 + a(t)^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin[\theta]^2\mathrm{d}\phi^2\right)\right), \end{aligned} \tag{1}$$

where $N(t)$ is a lapse function[5] and $a(t)$ is the **scale factor** that describes the size of the three-dimensional spatial slices. We will see later that the behaviour of $a(t)$ it is tightly constrained by any type of energy densities that the geometry may contain.

Before we start talking about the spatial properties of the previous line invariant (1), let us first discuss about the lapse function $N(t)$. This function is responsible for time reparametrisation invariance. As we do not want to overcomplicate our computation, the most useful and convenient choices for $N(t)$ are:

- $N(t) = 1$: This is the choice of *global time,* t. Any clock measuring this choice of time is moving along the Hubble flow, which is just the motion of astronomical objects just due to the expansion of the universe.

- $N(t) = a(t)$: This is the so-called *conformal time,* η. The conformal time is the amount of time it would take a photon to travel from where we are located at to the furthest observable distance.

The extra parameter we have not talked about yet is k. This has the power to change the topology of the spatial sections in the line invariant. It comes in three different flavours [6]:

- $k = 0$: A rather boring case. No curvature, where the spatial sections of the geometry are *flat*, like $\mathbb{R}^3$.

- $k = 1$: At this value, the spatial sections are *closed*, as in $\mathbb{S}^3$.

- $k = -1$: Spatial sections are *open*, as in the hyperboloid $\mathbb{H}^3$.

So the line invariant (1) allows us to describe a dynamical universe with different types of spatial curvature. What are we supposed to do with this tool? How do we get a specific equation(s) that explicitly describe the evolution of the universe in terms of its curvature and content? It is at this point where Einstein's equation comes in.

1.1 Friedmann equations

Einstein's equations are *Equation of Motions* (EOM) that can be used to describe the dynamics of the universe. These equations can be obtained by extremising the Einstein-Hilbert action

$$S[g_{MN}, \phi_i] = \int \mathrm{d}^D X\sqrt{|g|}\left(\frac{R^{(D)}}{2\kappa_D} + \mathscr{L}_{\mathrm{mat}}(\phi_i, \mathrm{d}\phi_i)\right), \tag{2}$$

with $\sqrt{|g|}$ the square root of the minus determinant of the metric, $\mathscr{L}_{\mathrm{mat}}$ as the matter lagrangian of some fields ϕ_i, coupled to gravity and R is the Ricci scalar, which carries geometrical information. κ_D encodes information about the D-dimensional Newton's gravitational constant as:[6]

$$\kappa_D = 8\pi G_D = M_{\mathrm{Pl}}^{2-D} = \ell_{\mathrm{Pl}}^{D-2}, \tag{3}$$

[5] A general function depending on time and other coordinates. The "duration" of the measured time intervals depends on the choice of this function.

[6] Note that we will work with $\hbar = c = 1$. This will always be the case unless otherwise specified.

where $M_{\rm Pl}$ and $\ell_{\rm Pl}$ are the Planck mass and length for D dimensions.[7] As we are working out the classical cosmology scenario, so far we will stick to $D=4$. To obtain the Einstein equations, we need only to vary the action (2) with respect to g_{MN} (g_{ab} now that we fix the dimensions to four) to obtain:

$$\underbrace{R_{ab}-\tfrac{1}{2}g_{ab}R}_{G_{ab}}=\kappa_4\underbrace{\left(\mathscr{L}_{\rm mat}\,g_{ab}-2\frac{\delta\mathscr{L}_{\rm mat}}{\delta g^{ab}}\right)}_{T_{ab}}=\kappa_4 T_{ab}. \tag{4}$$

The left-hand side of equation (4) condenses pure geometry information in the Einstein tensor G_{ab}, while the right-hand side represents the matter field contribution. This is packed into a rank two tensor called the *energy-momentum tensor* T_{ab}. As we assume that the universe is homogenous, isotropic and its contents can be described as a perfect fluid, the energy-momentum (EM) tensor takes the simple form:

$$T_{ab}=(\rho+p)\,u_a u_b+p\,g_{ab}, \tag{5}$$

where $u_a=(-N(t),0,0,0)$ is the fluid four-velocity, and (ρ,p) its energy density and pressure. These can be used to describe pressureless matter (dust) or relativistic one (radiation), among others. As energy is a conserved quantity, the EM tensor must also be conserved, i.e. $\nabla_a T^{ab}=0$.

From the clear relationship between geometry and matter content in Einstein equation (4), Wheeler once stated: *"Spacetime tells matter how to move; matter tells spacetime how to curve"*. If we choose $N(t)=1$ (i.e. global time coordinate) for the FLRW metric (1) described above, the Einstein's equation yields two equations[8] of the form:

$$\frac{\dot a^2}{a^2}=\frac{8\pi G_4}{3}\sum_i\rho_i-\frac{k}{a^2}, \tag{6}$$

$$\frac{\ddot a}{a}=-\frac{4\pi G_4}{3}\left(\sum_i\rho_i+3p_i\right), \tag{7}$$

where $\frac{\dot a}{a}$ is the *Hubble rate* H, which measures the expansion of the universe (static when $H=0$). These are the *Friedmann equations.*

The first Friedmann equation (6) describes how the expansion rate of the universe is governed by its content and topology, while the second equation (7) describes for its acceleration. It is easy to observe that if the universe is expanding faster every day, this implies that $\left(\sum_i\rho_i+3p_i\right)<0$ in Eq. (7). In order to obtain information about what may be responsible for this behaviour, we need to characterise different types of content. This is done by the *equation of state*, which relates pressure to energy density as $\omega=\frac{p}{\rho}$. Any type of matter with $\omega>-1/3$, will be responsible for any decelerated expansion of the universe. On the other hand, any content with $\omega<-1/3$ will accelerate the expansion.[9] We clearly observe that the universe is expanding in this era, but there is nothing that we have (yet) discovered that can be identified with this equation of state. As the universe is a dark place, and this unknown energy density seems to be the source of the accelerated expansion, it has been given the famous name of *dark energy*.[10] In general, the value of the state parameter for any energy density can be determined from basic principles such as:

[7] Planck scales makes refer to scales at which quantum effects of gravity become significant. When this energy scale is reached, gravity cannot be ignored in other fundamental interactions.

[8] One for the tt-component and three identical copies for the spatial ones.

[9] This follows from the Null Energy Condition (NEC) [15]. This condition states that the matter energy- momentum tensor T_{MN} obeys

$$T_{MN}n^N n^M>0, \tag{8}$$

for any null or light-like vector n^M , i.e. for any vector satisfying $g_{MN}n^M n^N=0$. This implies that ω is bounded as $-1\le\omega\le1$.

[10] The first appearance of the term "dark energy" can be found in [16].

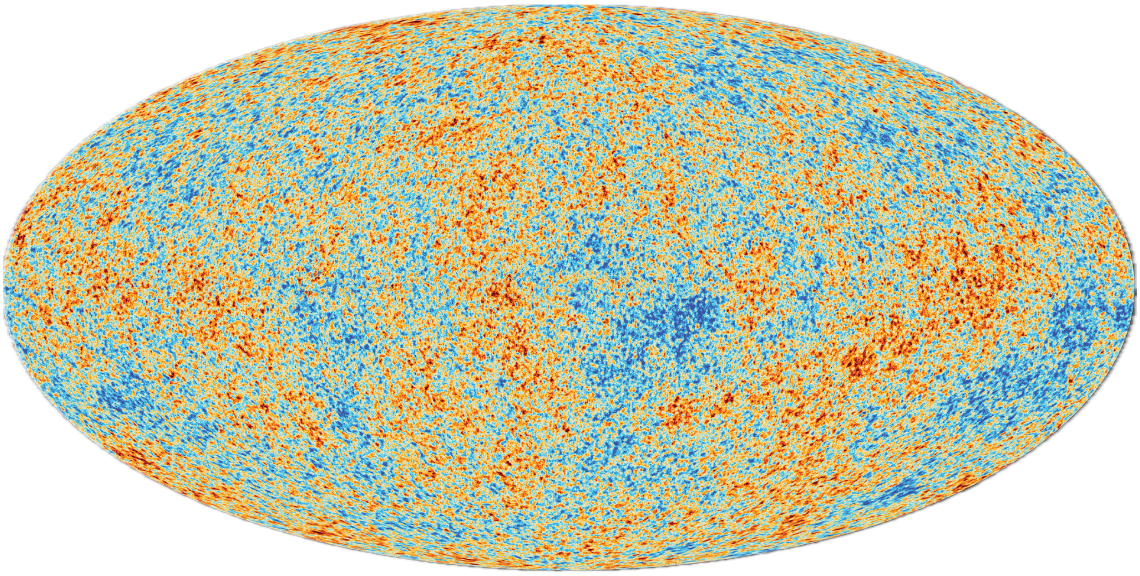


Figure 1: The Cosmic Microwave Background (CMB) [13,14]. This radiation background fills all of observable space. It is a remnant of the early universe, when it was filled with a hot-plasma of sub-atomic particles that did not allow photons to travel freely. When the plasma cooled down and neutral atoms could form, the scattering stopped and photons were released in the distribution observed above, carrying information about temperature fluctuations in the early era of the universe. The distribution of densities and the Hubble constant H_0 can be extracted from this imprinted background of the early universe. Credit: ESA/Planck Collaboration.

- In the case we consider non-relativistic matter, the energy density ρ_m is dominated by the rest mass energy $E = mc^2$, as the momentum, and hence, the exerted pressure are negligible compared to ρ_m. Thus, to a good approximation, $\omega \simeq 0$.
- For radiation, we are now dealing with relativistic effects. Following the same reasoning as before, the pressure, which is proportional to the velocity $v \simeq c$ will be relevant. Assuming an isotropic distribution of the pressure across the three spatial dimensions, this can be described by $\omega = 1/3$.
- Finally, dark energy seems to exert a *repulsive* force, that causes the universe to expand. This is described by a state parameter $\omega = -1$. This value can be constrained observationally by data extracted from *Barionic Accustic Oscillations* (BAO), perturbations of the hot plasma of the early universe that imprinted in the *Cosmic Microwave Background* (CMB). The most recent value (2018) is $\omega_\Lambda = -1.028 \pm 0.031$ [17].

Energy content	Scaling	Present value (Ω)
Matter	$\rho_M \sim \rho_{M_0} a^{-3}$	0.3103 ± 0.0057
Radiation	$\rho_\gamma \sim \rho_{\gamma_0} a^{-4}$	~ 0
Dark energy	$\rho_\Lambda \sim \rho_{\Lambda_0}$	0.6897 ± 0.0057
Curvature	−	0.0005 ± 0.0039

Figure 2: Observed Cosmological parameters [17]. Ω is the normalised fraction between ρ_i and $\rho_{\text{crit}} = 3H_0^2/8\pi G_4$. It is the sum of the normalised densities $\sum_i \Omega_i = 1$ that gives the famous distribution of the energy content of the universe: (Roughly) 70% of dark energy, 25% of dark matter and 5% of ordinary matter and radiation.

Equipped with this information, we are one step closer to understanding the true power of Friedmann's equations (6, 7). But one step at a time, cosmic hitchhikers! If the universe is evolving, so does the energy density of its contents and viceversa. This is governed by the conservation of energy. The explicit expression for the covariant derivative of the EM tensor (5) in a FRLW geometry (1) gives a relation between the energy density ρ_i of any content and the scale factor $a(t)$ of the form:

$$\rho \propto a^{-3(1+\omega)}, \tag{9}$$

One of the most interesting features of relation (9) is that ρ_Λ remains the same yesterday, today and tomorrow. It is not diluted by the expansion of spacetime. This can be interpreted as an intrinsic property of spacetime itself, which can be captured by an additional EM tensor of the form:

$$T_\Lambda = -\rho_\Lambda g_{ab} = -\frac{\Lambda}{\kappa_4} g_{ab}, \tag{10}$$

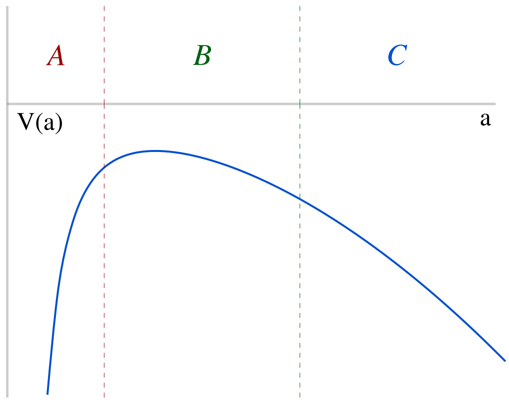
Where Λ is a constant of dimension $[\text{Length}]^{-2}$. This is the famous *cosmological constant* (CC),[11] which encodes the information of a non-varying energy density throughout the expanding cosmos. As this can be interpreted as an intrinsic geometrical property, it is perhaps more appropiate to move it to the LHS of Einstein's equation (4) as:

$$R_{ab} - \tfrac{1}{2} g_{ab} R + \Lambda g_{ab} = \kappa_4 T_{ab}, \tag{11}$$

which is the usual Einstein equation for cosmology. Now that we have fully identified all the characters in this cosmic play, let us come back to the Friedmann equation (6). Taking into account all possible types of energy densities ρ_i, we obtain the following expression:

$$\dot{a}^2 = \frac{8\pi G_4}{3}\left(\frac{\rho_{\text{mat}}}{a} + \frac{\rho_\gamma}{a^2} + \rho_\Lambda\, a^2 \underbrace{- \frac{3k}{8\pi G_4}}_{+\rho_c}\right), \tag{12}$$

where we have rewritten the curvature contribution in a more appropiate notation, i.e. ρ_c. A simple look at expression (12) is enough to see that it can be written in a quite familiar conservation equation as $T + V = 0$, where the potential $V(a)$ corresponds to the RHS of (12), with the opposite sign.



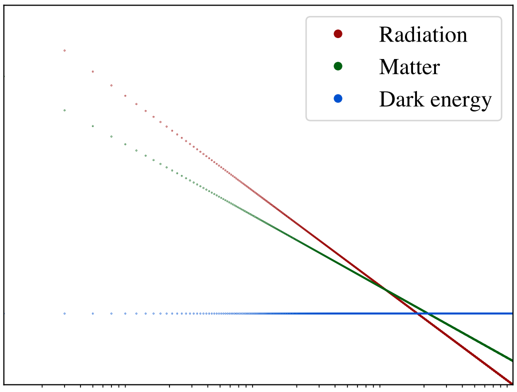


Figure 3: (Left) Qualitative behaviour of the effective potential in equation (12). Each part the potential $\{A, B, C\}$ corresponds to different eras dominated by radiation, matter and dark energy respectively. (Right) Evolution of the energy density of radiation, non-relativistic matter (dust) and dark energy as contents of the universe (logarithmic scales).

Our current observations suggest the fact that our universe has entered region C in figure 3 some years ago (about 4×10^9 years ago). We can also see that the gradient of this potential

[11] The famous one that *Albert* introduced in his equation to obtain a static universe.

in region C is negative, indicating an accelerated expansion of the universe, in agreement with observations. It is also important to note the divergence of the potential $V(a)$ when $a \to 0$, which causes a singularity at $t = 0$ (conventionally chosen [18]). What does this mean? From a classical cosmology point of view, this corresponds to the *Big bang*. A singular state that we **cannot understand** with the tools at hand, i.e general relativity and classical cosmology, where all the contents of the universe were compressed into a single point at extremely high temperatures [19]. This should already make us bat an eye. We are trying to apply a *classical* framework to a system whose energy regime very well requires a *quantum* description of all the forces involved. It should come as no surprise that we end up with singularities in our description if *we are using a sledgehammer to crack a nut.*

But we cannot be so conformist. If we cannot understand through the lens of a theory such as general relativity, this may indicate that our toolbox lacks some important tools.[12] Before we go to our favourite DYI (Do it yourself) store to buy these tools, we must first understand *what* and *how* we want to fix our lack of understanding. This will be the main topic covered in the next section 1.2.

1.2 Issues with the Λ-CDM Model

We talked about some limitations of the standard model of cosmology at the end of section 1.1. In this section we will look at them in more detail. Let us start with two of its most well known issues.

1.2.1 Horizon problem

To understand this fundamental problem which the standard model of cosmology cannot explain, we must first define what to be in *contact* is. There is nothing faster than light in the universe [20]. This imposes an upper bound on the size of a region where two points could have been in causal contact, i.e. information between the points could have travelled in a period of time less or equal to the time it would take light to travel the distance separating them. This region is called the *light cone*. Massless particles, as photons, follow null geodesic trajecto-

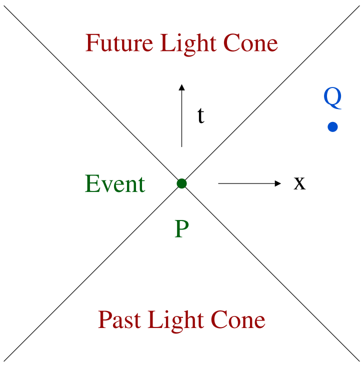


Figure 4: Light cone. Photons travel along world lines of zero proper time, $ds^2 = 0$, called null geodesics (both lines at 45 degrees). The set of all null geodesics passing through a given event P in spacetime is called the *light cone*. The interior of the light cone is defined as the region of spacetime causally related to that event P. Causally disconnected events (such as Q) are outside the lightcone related to the event P.

ries, which imply $ds^2 = 0$. Furthermore, as the universe is isotropic, only the radial coordinate will be relevant for computations. Calculating in conformal time η, we can extract the following

[12] The most prominent absence is that of a quantum description of gravity.

relationship from the light-like particle line invariant:

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}\chi^2\right] = 0 \longrightarrow \chi(\eta) = \eta_f - \eta_i = \int_{t_i}^{t_f} \frac{\mathrm{d}t}{a(t)}, \tag{13}$$

where χ is a generic radial coordinate that accounts for all possible topologies [21]. When $\eta_i = 0$, $\chi(\eta)$ represents the *comoving particle horizon*, which is the maximum distance light can travel between time 0 and aby other time t. One can then further rewrite expression (13) in terms of the *comoving Hubble radius* $R_H = (aH)^{-1}$, which is the radius of the observable universe at a given time t. Then:

$$\eta_f = \int_0^{t_f} \frac{dt}{a(t)} = \int_0^a \frac{da}{Ha^2} = \int_0^a d\ln(a)\frac{1}{aH}. \tag{14}$$

We would like to make contact with today's data, so we can rewrite the Hubble radius at any moment to today's by massaging the first Friedmann equation (6) together with the energy density evolution (9) to see that:

$$R_H = R_{H_0}\, a^{1/2(1+3\omega)}. \tag{15}$$

It is important to note here the dependence of the exponent, which depends on the state parameter ω. This leaves us with a dependence of η as:

$$\eta \propto a^{1/2(1+3\omega)}, \tag{16}$$

which means that the comoving horizon grows monotonically with time. This implies that some of the scales that are entering our horizon today were **outside** the Causal contact horizon in the past (The universe was dominated by radiation and matter, as can be seen in (3)). This can be seen in the CMB. Its homogeneity suggests that the universe was highly homogeneous at that time, but we have just seen that these homogenous regions could not have been in Causal contact in the past. How is this possible? What key concept is missing from our interpretations?

1.2.2 Flatness problem

Imagine that you are doing your laundry. Normally, when you take your clothes out of the washing machine, they are full of wrinkles. Sometimes they get even worse after they dry. Then, you patiently iron them down to smooth the surface. Now, try to imagine the opposite case; that your clothes were extremely smooth and flat when you took them out from the washing machine and after ironing, they are still smooth, but not as nicely wrinkle-free as they were just when you opened the washing machine door. Makes no sense, does it? Something similar is whatour universe shows.

Let us rewrite the first Friedmann's equation (6) as:

$$\left(\Omega(a)^{-1} - 1\right)\sum \rho_i\, a^2 = -\frac{3k}{8\pi G_4}, \tag{17}$$

where $\Omega(a) = \sum \rho_{0_i} a^{-3(1+\omega)}/\rho_{\text{crit}}$, with $\rho_{\text{crit}} = 3H^2/(8\pi G_4)$. Note that the RHS of the previous expression is a constant quantity, [13] while the LHS evolves with time. The density ρ will decrease with time, counterbalanced by the increase in the scale factor a. As the universe was dominated by radiation and matter in the past, this implies that the combination ρa^2 has decreased with time. In order to keep the behaviour constant, this requires that $\Omega(a)^{-1} - 1$ has increased in the same way. This constrains the value of $\Omega(a) \simeq 1$ further and further as we go back in time. This points to an extreme *fine-tuning* of $\Omega(a)$ at early times, to deviations from one of no more than $\mathcal{O}(10^{-16})$ during the nucleosinthesis period, for example. Any deviation greater than such ridiculously small values would not have allowed the flat universe that we observe today.

[13] In fact, 0, if we account that $\Omega_k \simeq 0$ as we saw in figure 2.

1.2.3 The Hubble tension

The Hubble constant today H_0, which is defined as the ratio between the speed and the distance of an object in space ($H_0 = v/d$), can be measured in two different ways:

- You can find the speed of the object (usually a galaxy) by looking at its redshift. For the distance, the most reliable way is to make use of *standard candles*; Type Ia supernovae whose peak luminosity is the same, no matter where they are in the universe. Given the relationship between relative and absolute magnitudes, it is possible to extract the distance to these objects with good accuracy. This method is used to study nearby celestial objects, i.e. not looking back much into the past. The current value for H_0 using this method is $H_0 = 72.3 \pm 1.3\,\mathrm{km\,s^{-1}Mpc^{-1}}$ [17].

- An alternative way of obtaining H_0 from observations with our current technology is to study the CMB, i.e. to look far into the past. Theoretical predictions for the CMB can be obtained by tuning certain parameters in the Λ-CDM model, such as curvature, energy densities, etc. These predictions can then be compared with the observations, to decide which one is the best fit. The best fit so far yields a H_0 value of $67.4 \pm 0.5\mathrm{kms^{-1}Mpc^{-1}}$ [22].

As we can see, there is an apparent tension between these two results which has been confirmed by technological development over the years.[14] What could be the reason for obtaining two different values for the expansion rate of the universe at two different eras? Several proposals have been discussed, without observational confirmation or agreement in the community yet (see [23]). In any case, observations point to an existing difference that cannot (yet) be explained by the standard model of cosmology.

1.2.4 The cosmological constant problem

We have previously talked about different observations of the Hubble constant using two different methods of measurement. However, this is not the only dichotomy that exists in Λ-CDM model.

Let us quantise matter fields appearing in the Einstein-Hilbert action (2) as proposed by Weinberg [24]. At the end of the day, the remaining physical interactions (electromagnetism, weak and strong nuclear forces) are quantised in the standard model of particle physics. Each of these interactions, has a *vacuum energy*, corresponding to the minimum background energy that space itself has. In the absence of curved spacetime, this vacuum energy can somehow be ignored. But, this is not the case if we want to study any of these fields in the presence of gravity. We know from the equivalence principle that gravity couples to all possible forms of energy, including that of the vacuum.

The quantisation of matter fields will result in an additional constant energy density contribution[15] to the energy-momentum tensor T_{ab}. As this term is constant and with the same dimensionality as the cosmological constant (i.e. $[\mathrm{Length}]^{-2}$), we can add it to Λ to obtain an *effective* cosmological constant as:

$$\Lambda_{\mathrm{eff}} = \Lambda + \underbrace{\kappa_4 \rho_{\mathrm{vac}}}_{\Lambda_{\mathrm{vac}}}, \tag{18}$$

which is in fact the cosmological constant that we are able to measure from observations. Let us now identify each of its contributions numerically. According to [17], the effective value of

[14] It could be that our technological limitations were the reason for the difference, but as technology has developed, this difference has been confirmed with better and more reliable measurements.

[15] This additional contribution comes from the closed-loop Feynmann diagrams (i.e. quantum fluctuations around the vacuum state) of each of the matter fields involved. We will not include a discussion of this computation in these notes, but we refer the reader to [25, 26] for further reading and details.

the cosmological constant is:

$$\Lambda_{\text{eff}} \sim 3 \times 10^{-122} M_{pl}^2, \tag{19}$$

while any first-order correction due to quantum fluctuations of the matter fields is:

$$\Lambda_{\text{vac}} \sim \frac{M_{\text{Pl}}^2}{16\pi^2} \sim 6.3 \times 10^{-3} M_{\text{Pl}}^2. \tag{20}$$

This is a difference of about 10^{120} orders of magnitude (in Planck units), as often stated in the literature. However, previous computations of the vacuum energy are slightly handwavy, because the renormalisation scheme does not respect Lorentz invariant. As noted in [25], an accurate renormalisation will return a mismatch of 54 orders of magnitude instead of 120. In both cases, this means that the bare cosmological constant Λ has to be extremely fine tuned to cancel out in such an almost perfect way that the small effective one we can observe remains. A recent discussion about this issue can be found in [27]. This last required coincidence is what gives title to this section.

The first two previously discussed problems of the Λ-CDM model (i.e. the horizon and flatness problems) may find possible solutions in the proposed inflation mechanism.[16] However, there is another underlying problem, which is problably related to all previously discussed problems, and which was already pointed out by Lemaître in [19]; General Relativity is a classical and effective[17] field theory of spacetime itself. This means that it cannot provide a description of gravity in the short-range regime. There is a cut-off, an upper bound, the Planck length ℓ_{Pl} at which quantum effects, hence new degrees of freedom, will become important. In this regime, the classical description of spacetime breaks down, pointing to the need for a more fundamental theory that is completed in the **ultraviolet** regime, i.e. a theory that is well defined at arbitrary high energies and without free parameters. A theory of **quantum** gravity. At this point the question arises:

How and where can one acquire such powerful tool as an UV-complete theory of gravity?

Perhaps it is time to leave the comfort of our cottage and explore the boundaries of our knowledge, in search of a clue that will bring us closer to the desired fully consistent theory of quantum gravity. It is well known that the best possible candidate we have so far is string theory. However, to explore such a beautiful theory is not the scope of these notes. Instead, it could be instructive to explore what the intricacies and subtleties of a quantised version of classical four-dimensional cosmology are.

2 Quantum Cosmology

The previous section was devoted to presenting the observable universe and its dynamics from a classical perspective. General relativity seemed to work like a charm in the usual regimes where the Λ-CDM can be applied. However, we have also seen in section 1.2 that it has its flaws. Without delving into too much detail, some of these issues can be solved assuming that the universe underwent a period of exponential expansion called *inflation* inmediately after the Big Bang [28,30–34]. Althought this proposal elegantly solves some of the Λ-CDM problems such as the horizon or flatness problem, it raises new complicated questions hard to answer.

[16] This mechanism proposes that the universe went through a de Sitter phase (i.e. a phase in which dark energy dominates the dynamics, giving rise to an exponential behaviour of the scale factor $a \sim e^{Ht}$) moments after the Big Bang. As these notes will be mainly oriented towards the understanding of late-time cosmologies, we will not touch much on the concept of inflation. However, we leave some references here for the curious reader [21,28–30].

[17] An effective field theory is one that provides simple a description of phenomena when applied at the right scale. It includes the appropriate number of degrees of freedom to describe physical phenomena occurring at a chosen length scale or energy scale, and ignores substructure and degrees of freedom that may occur at shorter distances.

For example, one may wonder what the initial conditions for any fields controlling such exponential enlargement were. This will undoubtedly lead us towards the *fine tuning* problem [35]. Moreover, it is still necessary to find reliable observational evidence which can prove that such process took place at the beginning [36, 37].

But there is an even more fundamental problem lying behind that of initial conditions. As Lemaître already pointed out in 1933 [12] and discussed in section 1, the effective potential divergence in Eq. (12) when $a \to 0$ signals a breakdown of our classical understanding of cosmology. The *primeval atom* proposed by the priest in [7] could be defined as a metastable and quantum state where the notion of spacetime ceases to exist. The key word to put the focus on here is *quantum*.

One of the main priorities of the physics community has been to find a formalism to describe all the fundamental interactions within the same framework. While electromagnetism and both the strong and weak nuclear forces can be studied within the framework of the standard model of particle physics [38, 39], gravity remains as a troublemaker to our wishes, unleashing infinities that cannot be tamed. The unified description of the three forces mentioned above can work out in energetic regimes where gravity is not strong enough to affect their interactions. However, this was not what the weather looked like at the very beginning of the universe ($\sim 10^{30}\,K$). In this regime, the Compton wavelength of a particle is more or less equal to its gravitational (Schwarzschild) radius. Hence, any quantum fluctuation would "blur" the classical concept of spacetime, pointing to a breakdown of the classical description of gravity.

Although the dream is to achieve a consistent theory of quantum gravity, the rentless efforts of the physics community have not yet provided the desired result. String theory [40–42] seems the best candidate we have for this title. However, these notes are oriented toward a more familiar four-dimensional approach to the cosmos. In this section, we will take a more modest approach than looking at string theory. Instead of quantising a non-perturbative renormalisable theory as gravity, we will try to describe the *whole* state of the universe with a semi-classical description, via the canonical quantisation of General Relativity. The premise is that this semi-classical approximation should coincide with the semi-classical low-energy description of the yet-to-be found theory of quantum gravity. This is the main aim of *Quantum* Cosmology [43, 44].

The idea is somehow simple: One takes one's favorite universe, described by the rules of General Relativity and proceeds to quantise canonically by following the Dirac's method [45] as if it was an usual quantum mechanics system. This implies identifying what the canonical variables are and to introduce a quantum wavefunction ψ (i.e. a quantum state $|\psi\rangle$ living in a Hilbert space) to represent the state of the universe. When the canonical variables and their conjugated counterparts have been promoted to operators, an Schrödinger-like equation can be defined to describe the evolution of the state of the universe. Finally, one would need to solve for the specific quantum state $|\psi\rangle$ that solves the aforamentioned equation. This requires us to provide the right set of boundary conditions. An interesting question to ask here would be: What is the right choice of boundary conditions? How can we define a "boundary" for the quantum system to be studied if we have never left it? We will see in this section that this still remains as a source of debate.

In this section we will not provide a complete review of the current state of quantum cosmology, but we will introduce the basic notions and framework that will be latter discussed (from a higher dimensional point of view) in this thesis. We will start with a discussion of the quantisation procedure for the most general four-dimensional cosmological configuration in subsection 2.1. We will then realise that the amount of information to be handled is overwhelming, which will require us to drastically reduce the number of variables controlling the system. A simple, yet powerful toy model will be considered in subsection 2.2, where we will also discuss about the physical implications of the most well-known boundary condition proposals.

2.1 Wheeler-DeWitt equation and superspace

Before we start with the contents of this section, we would like to invite our dear reader to enjoy the hypersurface discussion in these notes, so that the lecture of these will be a more pleasant experience after having acquired some familiarity with the to be used geometrical notation.

Let us start by choosing a Lorentzian manifold $\mathscr{M}$ that accepts a global time coordinate. This type of manifold always accepts time-orientability, which allows us to simplify the computation by decomposing the spacetime components. This will consist in separating the space slices from the global time coordinate t. Each spacelike hypersurface of constant time t will be denoted by Σ_t. This is is the ADM formalism, named after Arnowitt, Deser and Misner [46].

The set of coordinates used to describe the decomposition foliation is given by:

$$x^a = (t, x^i), \tag{21}$$

and the most general expression of the metric on the manifold $\mathscr{M}$ in these coordinates is:

$$\begin{aligned} \mathrm{d}s^2 &= g_{ab}\mathrm{d}x^a \mathrm{d}x^b \\ &= -\left(N^2 - N_i N^i\right)\mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}(t,x)\,\mathrm{d}x^i \mathrm{d}x^j, \end{aligned} \tag{22}$$

where $N \equiv N(t)$ represents the lapse function, as discussed in section 1. The function N_i is called the shift function, and measures the path difference between the same point p on the hypersurface Σ_t at different "slices" of time t. When $N_i = 0$, one recovers the usual description in comoving spatial coordinates. We will see that these two functions will play the role of constraints when we study the dynamics of the system. The metric h_{ij} represents the spatial sections of the four-dimensional geometry, i.e. the metric induced on them.

The dynamics of these slices will be controlled by the classical Einstein-Hilbert action (2) enhanced by the Gibbons-Hawking-York boundary term [47,48]. This extra piece accounts for any extrinsic contribution, i.e. how the the Σ_t slices is embedded in the whole four-dimensional space. The total action is given by:

$$S[g,h,\Phi] = \frac{1}{2\kappa_4}\int_{\mathscr{M}} \mathrm{d}^4x\sqrt{|g|}\left({}^{(4)}R - 2\Lambda_4\right) + \frac{\epsilon}{\kappa_4}\int_{\partial\mathscr{M}} \mathrm{d}^3x\sqrt{|h|}K + S_m[\Phi], \tag{23}$$

where $|h| = \det h_{ab}$ the determinant of the induced metric on the boundary $\partial\mathscr{M}$. ϵ represents the norm of the normal vector n_a and K is the trace of the extrinsic curvature (see this). Finally, any matter fields are encoded in the action term $S_m[\Phi] = S_m[\phi_0, \cdots \phi_n]$. This four-dimensional action can be broken down into its $3+1$ slice decomposition. One can decompose the Ricci tensor using this and write:

$$S[h,\Phi] = \int \mathrm{d}L = \frac{1}{2\kappa_4}\int_{\mathscr{M}} \mathrm{d}t\,\mathrm{d}^3x\,\sqrt{|h|}\,N\left({}^{(3)}R - K_{ij}K^{ij} - K^2 - 2\Lambda\right) + S_m[\Phi], \tag{24}$$

with the extrinsic curvature explicitely given by:

$$K_{ij} = \frac{1}{2N}\left(\partial_t h_{ij} + \nabla_i N_j - \nabla_j N_i\right). \tag{25}$$

Although it may seem appealing to compute the equations of motion directly from the action (24), it will be more illustrative to perform a Legendre transformation to the Lagrangian and obtain the Hamiltonian, as proposed by the ADM formalism. This requires us to identify the canonical coordinates[18] in the system: $\{h_{ij}, N, N_i, \Phi\}$. Hence, the canonical momenta can be

[18]Note that the choice of canonical variables is not unique. Different choices will lead to different quantum theories upon quantisation. Here we will stick to the ADM choice [46].

computed in the standard way [49]:

$$\pi_{ij} = \frac{\delta L}{\delta \dot{h}_{ij}} = -\frac{\sqrt{|h|}}{2\kappa_4}\left(K_{ij} - h_{ij}K\right), \qquad \pi_i = \frac{\delta L}{\partial \dot{N}_i} = 0,$$
$$\pi_{\phi_n} = \frac{\delta L}{\delta \dot{\phi}_n} = \frac{\sqrt{|h|}}{N}\left(\dot{\phi}_n - N^i\partial_i\phi_n\right), \qquad \pi_N = \frac{\delta L}{\partial \dot{N}} = 0. \tag{26}$$

Note that the conjugated momenta associated to the lapse N and shift N_i functions are zero. This implies that we are dealing with Dirac's primary constraints [50]. Perhaps our reader has never heard of such constraints. Let us rephrase them in a more "peasant" language. To do this, we then write the Hamiltonian as:

$$S = \int \mathrm{d}t\,\mathrm{d}^3x\left(\pi_N\dot{N} + \pi^i\dot{N}_i - N\mathscr{H} - N_i\mathscr{H}^i\right), \tag{27}$$

where $\mathscr{H}_m$ represents the Hamiltonian piece for the matter fields ϕ_i and

$$\begin{aligned}\mathscr{H} &= 2\kappa_4 G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{2\kappa_4}\left({}^{(3)}R - 2\Lambda_4\right) + \mathscr{H}_m,\\ \mathscr{H}_i &= -2\nabla^j\pi_{ij} + \partial_i\mathscr{H}_m.\end{aligned} \tag{28}$$

Note that the derivatives of Eq. (27) with respect to the lapse N and shift N_i act as *Lagrange multipliers*, which will result in severe constraints as:

$$\mathscr{H}_i = 0, \qquad\qquad \mathscr{H} = 0. \tag{29}$$

From now on, we will refer to $\mathscr{H}$ as *The* Hamiltonian. This Hamiltonian will govern the evolution of the state of the universe along the space of configurations it can have. In order to have a good notion of distances and the geometry of such territory, we will define G_{ijkl}. This object receives the name of DeWitt metric [51] and it is formulated as

$$G_{ijkl} = \frac{1}{2}h^{-1/2}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right), \tag{30}$$

which characterises the geometry of the *superspace*. Formally, one can define this space by:

$$S(\Sigma) \equiv \left\{h_{ij}(x), \phi(x) \mid x \in \Sigma\right\} / \,\mathrm{Diff}_0(\Sigma). \tag{31}$$

As mentioned above, the *superspace* contains all possible different metrics h_{ij} and matter field configurations that the universe can have. It is infinitely dimensional, with a finite number of coordinates $\{h_{ij}(x), \Phi(x)\}$ at each point x of the three-dimensional surface Σ.

As we know from our quantum mechanics course, the quantum state of a system is represented by the wave function ψ associated to it [52]. This object is a functional $\psi[h_{ij}, \Phi]$ of the superspace, which provides the amplitude to find a particular hypersurface Σ_t of the universe with a given three-dimensional metric h_{ij} and matter field configuration Φ. Note the absence of any explicit time dependence. This should not be a cause of concern; At the end of the day, we are aiming to quantise general relativity, so there is an implicit time dependence in the spatial h_{ij} information.

Let us now quantise the system. According to Dirac's quantisation procedure [53], substituting the canonical momenta (26) by operators[19]

$$\pi^{ij} = -i\frac{\delta}{\delta h_{ij}}, \qquad\qquad \pi_n = -i\frac{\delta}{\delta \phi_n}, \tag{32}$$

[19]Note that we have neither π_i nor π_N. This is because the to-be operator version of the constraints should act like them, which is the case of the lapse and shift functions. This implies that the wavefunction ψ is independent of them.

which yields the following equations for ψ:

$$\mathcal{H}_i\psi = \mathcal{H}\psi = 0. \tag{33}$$

The first constraint forces the wavefunction ψ to be invariant under any three-dimensional diffeomorphisms. This will not be of much relevance in this work, as we will restrict ourselves to *comoving* frame (So $N_i = 0$). For the second constraint we especifically have:

$$\left[2\kappa_4 G_{ijkl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \frac{\sqrt{h}}{2\kappa_4}\left({}^{(3)}R - 2\Lambda\right) - \mathcal{H}_m\right]\psi = 0. \tag{34}$$

This equation is known as the *Wheeler-De Witt* equation [51] and it will be the main object of study in quantum cosmology. It describes the dynamical evolution of the wavefunction of the universe, hence its state. It also ensures the explicit **time independence** of the "timeless" wavefunction ψ. This equation can then be thought of a zero-energy analogue of the Schrödinger equation [54] due to its similarities. At the end of the day, it describes the temporal evolution of a quantum system.

We would not like to finish this section of the section without commenting on one of the most studied and reliable forms of solving Schrödinger-like equations in quantum mechanics: The path integral [38,55]. This object provides the probability amplitude for a system (i.e. a particle or a universe) to move between two different states within a give time interval. The trajectory it will follow will not be deterministic, as the uncertainty principle will limit our precision in calculating pairs of physical quantities [56]. In this case, we are required then to sum over *all* possible configurations that interpolate between the final and initial states of study. Morever, as our object of study is a gravitational system with potential topological changes,[20] this would require us to account for them along the path. This can be cast as:

$$Z = \left\langle h_{ij}^{(\mathrm{f})}, \phi_{\mathrm{f}}; \Sigma_{\mathrm{f}} \middle| h_{ij}^{(\mathrm{i})}, \phi_{\mathrm{i}}; \Sigma_{\mathrm{i}} \right\rangle = \sum_m \int \mathcal{D}g\mathcal{D}\phi e^{iS[g,\phi]}, \tag{35}$$

where the sum $\sum_m$ takes into account all possible topologies that the four-dimensional geometry can have. The integration is performed over all possible g_{ab} and ϕ_i configurations, represented by $\{\mathcal{D}g, \mathcal{D}\phi\}$. The action $S[g,\phi]$ is that described in Eq. (23). Note that this would imply a strongly oscillating integrand which could suffer from convergence issues when integrating. One might, in principle, think that an analytic continuation to the Euclidean description (i.e. $r = it$) would tame such a problem. Nothing further from reality. Divergences will continue appearing due to the non-renormalisable nature of gravity [57–59]. Furthermore, the non-perturbative aspect of this force will lead to an unbounded from below action (23) [60,61].

All in all, despite the difficulties, the path integral of gravity has been proved to be an extremely useful tool in the semi-classical (i.e. the quantum cosmology) approximation. In this regime, the path integral is a weighted sum over all solutions that extremise the action (23). This ease the computation, and allows us to define the wavefunction ψ describing the state of the universe as:

$$\psi[h_{ij}, \phi; \Sigma] = \int \mathcal{D}g\mathcal{D}\phi\, e^{i\,S[g,\phi]}. \tag{36}$$

This general wavefunction ψ satisfies the Wheeler-DeWitt equation (34). However,there is a subtle catch here; the path integral formalism does not provide a specific initial configuration state $|h_{ij}, \phi; \Sigma\rangle$. This brings us back to the initial condition issue discussed at the beginning of this section. We have solved for the most general solution of the wavefunction (36) and in order to select the *specific* wave that describes the evolution of the universe, we need to impose a set of boundary conditions on the countour of integration. From the persepective of a quantum

[20] Recall that we consider *all* possible intermediate states.

mechanics course, this is easy. You are given a potential with some boundary conditions, impose them and pick out the solution. As an external observer of the system, you have an idea of the "shape" of the studied system.[21] Nevertheless, within the framework of quantum cosmology, where the observer is part of the system, the choice of boundary conditions to be imposed is not so clear. Ideally, such a choice should be provided by the physics of the system. However, from a four-dimensional quantum cosmology point of view proposals and debates about the choice of boundary conditions are all we have to work with.

Before embarking ourselves on the study of the two most common proposals for the aforementioned discussion, let us first drastically reduce the number of degrees of freedom and limit ourselves to a reduced set of them to have a *concrete* description of a simple wavefunction describing the evolution of the universe. Then, we will be able to easily impose the two different boundary choices and delve into their physical implications.

2.2 Minisuperspace of quantum cosmology

Let us start by simplifying our superspace. Its infinite dimensionality does not help with computations. So the best way to proceed is to do what physicists do best; to approximate the system with a toy model. In this case, we will restrict our attention to just a few individual variables of the superspace and freeze any other degrees of freedom. The resulting configuration of the superspace is called *minisuperspace*.[22] This simplification will allow us to have a *tractable* set of degrees of freedom, which will facilitate any explicit computation.

The toy model that we have proposed considers the quantisation of an empty four-dimensional universe, with closed spatial section and a positive cosmological constant Λ_4. As we saw in section 1, this can be described by a Friedmann-Robertson-Leimatre-Walker metric (1) with $k = 1$. As our aim is to obtain the dynamics controlling its evolution, we need to substitute Eq. (1) in the action (23) to obtain the Lagrangian:

$$S = \frac{\mathrm{Vol}_{S^3}}{\kappa_4} \int \mathrm{d}t N \left(-\frac{3a\dot{a}^2}{N^2} + 3a - \Lambda_4 a^3 \right), \tag{37}$$

where $\mathrm{Vol}_{S^3} = 2\pi^2$ appears after integrating over the closed spatial directions $x = \{\alpha, \beta, \gamma\}$. Note that the only dynamical variable present in the previous action is the scale factor a. This implies that we have reduced the minisuperspace to only one dimension. The canonical coordinates of the Lagrangian (37) are then given by (a, π_a), where π_a is the conjugated momentum as

$$\pi_a = \frac{\delta L}{\delta \dot{a}} = -\frac{6\mathrm{Vol}_{S^3}}{\kappa_4 N} a\dot{a}. \tag{38}$$

The corresponding Legendre transformation will yield *The* classical Hamiltonian for this toy model, which is:

$$\mathcal{H} = -\frac{\kappa_4}{\mathrm{Vol}_{S^3}} \frac{\pi_a^2}{12\,a} + \frac{\mathrm{Vol}_{S^3}}{\kappa_4}\, a \left(\Lambda_4 a^2 - 3 \right). \tag{39}$$

If we now quantise the system as described in section 2.1, we need to replace the conjugated momentum π_a by $-i\partial_a$ and *The* Hamiltonian constraint $\mathcal{H} = 0$ by the Wheeler-DeWitt equation (34). Simplifying and rearranging terms so that the above equation resembles that of Schrödinger with an effective potential $V(a)$, we get:

$$\left[-\frac{1}{2}\frac{\partial^2}{\partial a^2} + \frac{\mathrm{Vol}_{S^3}^2}{\kappa_4^2} \underbrace{\left(6\,a^2 \left(3 - \Lambda_4\, a^2 \right) \right)}_{V(a)} \right] \psi_{4D} = 0. \tag{40}$$

[21] Another way of thinking about this is to try to explain the concept of phase transition from gas to liquid, but when the only conceptual physical understanding available is that of the liquid phase.

[22] No, physicists are not the best at naming things.

Note that the effective potential in figure 5 has two roots at $a_0 = 0$ and $a_* = 3/\sqrt{\Lambda_4}$. Returning to the Schrödinger equation analogy, we can think of our cosmos system as being driven by the effective potential $V(a)$, which has two clear regions separated by the *turning point* a_* when $\Lambda_4 > 0$. These would be called *quantum* region when $V(a) > 0$ and the *classical* region when $V(a) < 0$.

Given the "tameness" of the potential $V(a)$ in the Wheeler-DeWitt equation (40), the wavefunction solution can be found using the semi-classical Wentzel-Kramers-Brillouin (WKB) approximation [54]. For simplicity of notation, let us define:

$$S(a_f, a_i) = \frac{\mathrm{Vol}_{S^3}}{\kappa_4} \int_{a_i}^{a_f} \mathrm{d}a' \sqrt{2|V(a')|}, \tag{41}$$

which is the argument of the exponents in the wavefunction solution:

$$\psi(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} \mathscr{A} e^{S(a,0)} + \mathscr{B} e^{-S(a,0)} & \text{if } a < a_*, \\ \mathscr{C} e^{iS(a,a_*)} + \mathscr{D} e^{-iS(a,a_*)}, & \text{if } a > a_*. \end{cases} \tag{42}$$

The pairs $\{\mathscr{A}, \mathscr{B}\}$ and $\{\mathscr{C}, \mathscr{D}\} \in \mathbb{C}$ and can be related by the WKB formulas as:

$$\begin{cases} \mathscr{A} = \frac{1}{2} e^{-S(a,0)} \left(\mathscr{C} e^{i\frac{\pi}{4}} + \mathscr{D} e^{-i\frac{\pi}{4}} \right) \\ \mathscr{B} = e^{S(a,0)} \left(\mathscr{C} e^{-i\frac{\pi}{4}} + \mathscr{D} e^{i\frac{\pi}{4}} \right) \end{cases} \qquad \begin{cases} \mathscr{C} = \frac{1}{2} \mathscr{B} e^{-S_0 + i\frac{\pi}{4}} + \mathscr{A} e^{S_0 - i\frac{\pi}{4}} \\ \mathscr{D} = \frac{1}{2} \mathscr{B} e^{-S_0 - i\frac{\pi}{4}} + \mathscr{A} e^{S_0 + i\frac{\pi}{4}} \end{cases} \tag{43}$$

Note that the wavefunction Eq. (42) and its undetermined coefficients $\{\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}\}$ clearly point to those infinitely many possible solutions to the Wheeler-DeWitt Eq. (40). Here we will use the two most common boundary condition proposals introduced in section 2.1.

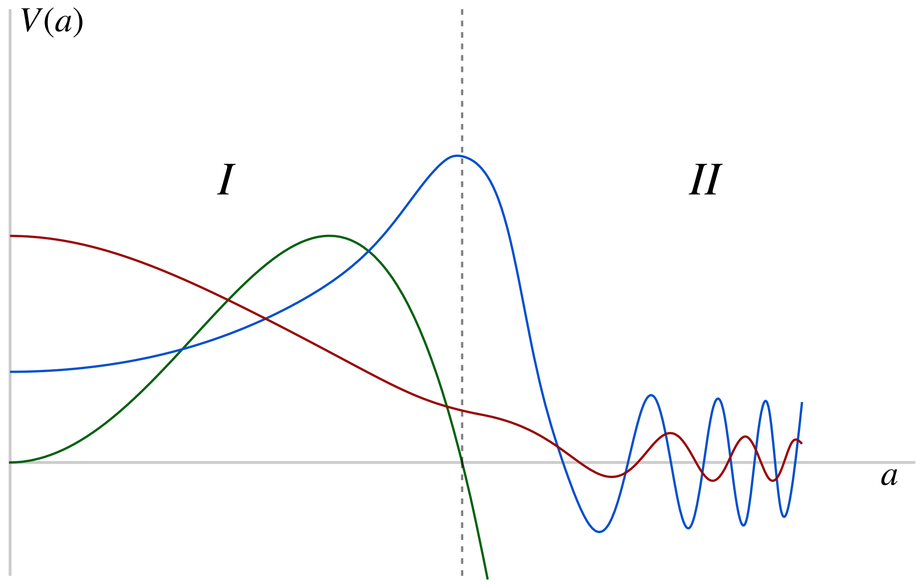


Figure 5: Plot of the potential controlling the dynamics of the wavefunction describing the evolution of the universe. The real parts of the Vilenkin and Hartle-Hawking wavefunctions are also shown. Region I corresponds to the quantum or Euclidean region, while Region II represents the classical part of the potential.

2.2.1 No-boundary proposal

This proposal, argued by Hartle and Hawking [62, 63], suggests that the Euclidean version of the wavefunction (36) should be restricted to the integration on compact four-dimensional

Euclidean manifolds. This implies that the slice Σ where ψ is defined is the **only** boundary to the system. From a more physical perspective, this can be translated into the claim that the universe had **no** singular **boundary** in the past. Hence the name of the proposal.

The interpretation of the no-boundary wavefunction is that the geometry arises from nothing. Translated this to the initial condition problem this would lead to conditions on $h_{ij}(x)$ and $\phi(x)$ and its derivatives in the imaginary time component. The full discussion of how to fix these restrictions can be found in [44]. This implies a choice of the coefficients for region I to be such that only the increasing exponential part of the wave exists, i.e. $(\mathscr{A},\mathscr{B}) = (1,0)$. Making use of the relations (43) one can also obtain the parametrical dependence in region II. The entire wavefunction (42) under the no-boundary condition proposal reads:

$$\psi_{\mathrm{HH}}(a) = \frac{1}{|V(a)|^{1/4}} \begin{cases} e^{S(a,0)} & \text{Region I} \\ 2e^{S(a_*,0)} \cos\left(S(a,a_*) - \frac{\pi}{4}\right) & \text{Region II} \end{cases}. \tag{44}$$

2.2.2 *Tunneling proposal*

The second well-known boundary proposal is that of Vilenkin [64–66]. This proposal requires the wavefunction ψ to be everywhere bounded, and at singular boundaries of superspace, ψ includes only outgoing modes. This can be thought of as a analogy to quantum tunneling in quantum mechanics. The boundary condition imposed there is an statement about outgoing modes at ∞. From a more physical point of view, the idea behind this proposal is that any possible state described by ψ should not include universe's states contracting down from an infinite size, i.e. only expanding states from "nothing". Given the simplicity of the minisuperspace and behaviour of the wavefunction solution in the classical region, is easy to see that $(\mathscr{C},\mathscr{D}) = (0,1)$. Using relations (43), and imposing $\mathscr{A} \sim 0$, as it is exponentially supressed, we find:

$$\psi_{\mathrm{V}}(a) \approx \frac{1}{|V(R)|^{1/4}} \begin{cases} e^{S(a_*,0)} e^{-S(a,0)+i\frac{\pi}{4}} & \text{Region I} \\ e^{-iS(a,a_*)} & \text{Region II} \end{cases}, \tag{45}$$

The explicit form of both the Hartle-Hawking and Vilenkin wavefunction expressions (44) and (45) allow us to extract what the nucleation probability of a universe with a cosmological constant Λ_4 is. This is no more than the amplitude of the wave under the *quantum* region as:

$$P_{HH} \propto \exp\left(+\frac{24\pi^2}{\kappa_4 \Lambda_4}\right), \qquad P_V \propto \exp\left(-\frac{24\pi^2}{\kappa_4 \Lambda_4}\right). \tag{46}$$

Note how the Hartle-Hawking probability favours the nucleation of universes with small positive cosmological constant Λ_4, while the opposite is true for the Vilenkin case. However, the question remains; what is the best set of boundary conditions to describe the beginning of our cosmos? Recent developments and claims [67] and counterclaims [68] have been made in the last years, making progress and shedding light on this long-standing problem. However, it could be that this is not enough. It could be that UV-complete theories, like string theory, may have the last word in the apparent choice of boundary conditions that the approach of quantum cosmology may have. And this would very well require another set of notes in their own.

References

[1] E. Hubble, *A relation between distance and radial velocity among extra-galactic nebulae*, *Proceedings of the National Academy of Sciences* **15** (1929) 168.

[2] P.J.E. Peebles and B. Ratra, *The cosmological constant and dark energy*, *Reviews of Modern Physics* **75** (2003) 559–606.

[3] N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini et al., *Planck 2018 results-vi. cosmological parameters*, *Astronomy & Astrophysics* **641** (2020) A6.

[4] A. Einstein, *Zur allgemeinen relativitätstheorie*, *Sitzungsberichte* **47** (1916) 214 .

[5] A. Einstein, *Die grundlage der allgemeinen relativitätstheorie*, *Annalen der Physik* **354** (1916) 769.

[6] A. Friedmann, *Über die krümmung des raumes*, *Z. Phys.* **10** (1922) 377.

[7] G. Lemaître, *A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae*, *mnras* **91** (1931) .

[8] P.J.E. Peebles, *Tests of Cosmological Models Constrained by Inflation*, *Astrophys. J.* **284** (1984) 439.

[9] S.M. Carroll, *The cosmological constant*, *Living Reviews in Relativity* **4** (2001) .

[10] V.C. Rubin and W.K. Ford, Jr., *Rotation of the Andromeda Nebula from a Spectroscopic Survey of Emission Regions*, *Astrophys. J.* **159** (1970) 379.

[11] V.C. Rubin, N. Thonnard and W.K. Ford, Jr., *Rotational properties of 21 SC galaxies with a large range of luminosities and radii, from NGC 4605 /R = 4kpc/ to UGC 2885 /R = 122 kpc/*, *Astrophys. J.* **238** (1980) 471.

[12] G. Lemaître, *Republication of: A homogeneous universe of constant mass and increasing radius accounting for the radial velocity of extra-galactic nebulae*, *General Relativity and Gravitation* **45** (2013) 1635.

[13] R.A. Alpher, H. Bethe and G. Gamow, *The origin of chemical elements*, *Phys. Rev.* **73** (1948) 803.

[14] A.A. Penzias and R.G. Wilson, *A measurement of excess antenna temperature at 4080-mc/s*, *The Astrophysical Journal* **142** (1965) 419.

[15] V.A. Rubakov, *The null energy condition and its violation*, *Physics-Uspekhi* **57** (2014) 128–142.

[16] D. Huterer and M.S. Turner, *Prospects for probing the dark energy via supernova distance measurements*, *Physical Review D* **60** (1999) .

[17] N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini et al., *Planck 2018 results-vi. cosmological parameters*, *Astronomy & Astrophysics* **641** (2020) A6.

[18] S.W. Hawking and R. Penrose, *The Singularities of gravitational collapse and cosmology*, *Proc. Roy. Soc. Lond. A* **314** (1970) 529.

[19] G. Lemaître, *Republication of: The beginning of the world from the point of view of quantum theory*, *General Relativity and Gravitation* **43** (2011) 2929.

[20] S.M. Carroll, *Spacetime and geometry*, Cambridge University Press (2019).

[21] D. Baumann, *Tasi lectures on inflation*, 0907.5424.

[22] A.G. Riess, S. Casertano, W. Yuan, J.B. Bowers, L. Macri, J.C. Zinn et al., *Cosmic distances calibrated to 1% precision with gaia EDR3 parallaxes and hubble space telescope photometry of 75 milky way cepheids confirm tension with ΛCDM*, The Astrophysical Journal Letters **908** (2021) L6.

[23] E.D. Valentino, O. Mena, S. Pan, L. Visinelli, W. Yang, A. Melchiorri et al., *In the realm of the hubble tension—a review of solutions**, Classical and Quantum Gravity **38** (2021) 153001.

[24] S. Weinberg, *The cosmological constant problem*, Rev. Mod. Phys. **61** (1989) 1.

[25] J. Martin, *Everything you always wanted to know about the cosmological constant problem (but were afraid to ask)*, Comptes Rendus Physique **13** (2012) 566.

[26] A. Padilla, *Lectures on the cosmological constant problem*, 1502.05296.

[27] U. Danielsson, *The quantum swampland*, Journal of High Energy Physics **2019** (2019) [1809.04512].

[28] A.H. Guth, *Inflationary universe: A possible solution to the horizon and flatness problems*, Phys. Rev. D **23** (1981) 347.

[29] D. Baumann and L. McAllister, *Inflation and string theory*, 1404.2601.

[30] J. Martin, C. Ringeval and V. Vennin, *Encyclopaedia inflationaris*, 1303.3787.

[31] A.R. Liddle, *An introduction to cosmological inflation*, astro-ph/9901124.

[32] A. Linde, *A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems*, Physics Letters B **108** (1982) 389.

[33] A. Linde, *Chaotic inflation*, Physics Letters B **129** (1983) 177.

[34] A. Starobinsky, *A new type of isotropic cosmological models without singularity*, Physics Letters B **91** (1980) 99.

[35] K. Landsman, *The fine-tuning argument*, 1505.05359.

[36] A. Ijjas, P.J. Steinhardt and A. Loeb, *Inflationary paradigm in trouble after planck2013*, Physics Letters B **723** (2013) 261.

[37] A. Linde, *Inflationary cosmology after planck 2013*, 1402.0526.

[38] M. Srednicki, *Quantum Field Theory*, Cambridge University Press (2007).

[39] M. Schwartz, *Quantum Field Theory and the Standard Model*, Quantum Field Theory and the Standard Model, Cambridge University Press (2014).

[40] J. Polchinski, *String Theory*, Cambridge Monographs on Mathematical Physics, Cambridge University Press (1998).

[41] K. Becker, M. Becker and J.H. Schwarz, *String theory and M-theory: A modern introduction*, Cambridge university press (2006).

[42] D. Tong, *String Theory*, 0908.0333.

[43] D.L. Wiltshire, *An introduction to quantum cosmology*, gr-qc/0101003.

[44] J.J. Halliwell and J. Louko, *Steepest Descent Contours in the Path Integral Approach to Quantum Cosmology. 1. The De Sitter Minisuperspace Model*, *Phys. Rev. D* **39** (1989) 2206.

[45] P.A.M. Dirac, *The fundamental equations of quantum mechanics*, *Proceedings of The Royal Society A: Mathematical, Physical and Engineering Sciences* **109** (1925) 642.

[46] R. Arnowitt, S. Deser and C.W. Misner, *Republication of: The dynamics of general relativity*, *General Relativity and Gravitation* **40** (2008) 1997–2027.

[47] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev. D* **15** (1977) 2752.

[48] J.W. York, *Role of conformal three-geometry in the dynamics of gravitation*, *Phys. Rev. Lett.* **28** (1972) 1082.

[49] H. Goldstein and V. Twersky, *Classical Mechanics*, *Physics Today* **5** (1952) 19.

[50] A. Hanson, T. Regge and C. Teitelboim, *Constrained hamiltonian systems*, Accademia Nazionale dei Lincei (1976).

[51] B.S. DeWitt, *Quantum theory of gravity. i. the canonical theory*, *Phys. Rev.* **160** (1967) 1113.

[52] J.J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, Cambridge University Press, 3 ed. (2020).

[53] P.A.M. Dirac, *The theory of gravitation in hamiltonian form*, *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* **246** (1958) 333.

[54] D.J. Griffiths and D.F. Schroeter, *Introduction to Quantum Mechanics*, Cambridge University Press, 3 ed. (2018).

[55] R. Feynman, *Feynman lectures on gravitation*, CRC press (2018).

[56] W. Heisenberg, *Über den anschaulichen inhalt der quantentheoretischen kinematik und mechanik*, *Zeitschrift für Physik* **43** (1927) 172.

[57] A. Shomer, *A pedagogical explanation for the non-renormalizability of gravity*, 0709.3555.

[58] M.H. Goroff and A. Sagnotti, *The ultraviolet behavior of einstein gravity*, *Nuclear Physics B* **266** (1986) 709.

[59] J. Doboszewski and N. Linnemann, *How not to establish the non-renormalizability of gravity*, *Foundations of Physics* **48** (2018) 237–252.

[60] G. Gibbons, S. Hawking and M. Perry, *Path integrals and the indefiniteness of the gravitational action*, *Nuclear Physics B* **138** (1978) 141.

[61] P.O. Mazur and E. Mottola, *The path integral measure, conformal factor problem and stability of the ground state of quantum gravity*, *Nuclear Physics B* **341** (1990) 187.

[62] J.B. Hartle and S.W. Hawking, *Wave function of the universe*, *Phys. Rev. D* **28** (1983) 2960.

[63] S. Hawking, *The quantum state of the universe*, *Nuclear Physics B* **239** (1984) 257.

[64] A. Vilenkin, *Creation of universes from nothing*, *Physics Letters B* **117** (1982) 25.

[65] A. Vilenkin, *Quantum creation of universes*, *Phys. Rev. D* **30** (1984) 509.

[66] A. Vilenkin, *Quantum cosmology and the initial state of the universe*, *Phys. Rev. D* **37** (1988) 888.

[67] J. Feldbrugge, J.-L. Lehners and N. Turok, *Lorentzian quantum cosmology*, *Physical Review D* **95** (2017) .

[68] A. Di Tucci and J.-L. Lehners, *No-boundary proposal as a path integral with robin boundary conditions*, *Physical Review Letters* **122** (2019) [1903.06757].