CdL and BT Instantons

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1 Introduction

We will review two main types of instantons[1] in these notes: Coleman-de Lucia [1] and Brown-Teitelboim [2] ones.

2 Coleman-De Lucia instantons

Let us consider a theory with a single scalar field $\phi(t,\bar{x})$ which is controlled by a potential $V(\phi)$ in a D-dimensional spacetime. We further assume that this potential has *two* non-degenerate minima $V_{\pm}$, such that $V_{+}(\phi) > V_{-}(\phi)$. The shape of this potential is sketched in figure 1. The action governing the dynamics of such configuration is:

$$S = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M\phi\partial^M\phi - V(\phi)\right), \tag{1}$$

where M goes from 0 to $D-1$. From a classical point of view, if the scalar field ϕ is at rest in the local minimum V_+, it will not have enough energy to climb over the potential barrier and reach the global minimum V_-. However, if quantum mechanics is taken into account, there is a probability that the field will "tunnel" through the barrier and end up in the global minimum V_-. In this framework, we will say that the local minimum V_- is *metastable* and a *decay* can take place, so that the field configuration "traverses" from the *false* vacuum V_+ to the *true* vacuum V_-.

The process described above was studied by Sidney Coleman and Curtis Callan in [3, 4]. The underlying physics of these decays is a first order transition[2] and it can be easily understood by a thermodynamic analogy.

[1] Instantons are solutions to the classical Euclidean equations of motion which interpolate between real classical motions of the system, and thus provide a semiclassical "path" by which the system tunnels from one classical regime to the other.

[2] i.e. the amount of energy absorbed or released by the system is fixed.

Consider a fluid, that is heated homogenously up to the point where it can start boiling. In its initial liquid phase, as much as you can try to have an even distribution of the temperature, there may be thermodynamic fluctuations at some given points where the temperature is slightly different from the surrounding volume. If this variation is favourable towards the phase transition temperature, a *bubble* of vapour phase will appear. This bubble can have two different fates: If its size is smaller that a certain threshold, so that the gain in energy density (i.e. the energy stored inside the bubble volume) is overcompensated by the loss of surface energy, then the bubble will collapse to nothing. On the contrary, a large enough bubble will have a favourable energy balance, which will cause the bubble to expand until all the liquid has undergone the phase transition to vapour.

This is a similar situation to the vacuum decay described some paragraphs above. We are now faced with a field decorating an empty spacetime. The configuration of the field is such that it starts in its *false* vacuum. In this case, not thermodynamical, but quantum fluctuations of the vacuum can occur, triggering the phase transition towards the *true* configuration in some specific region. This will happen through the nucleation of a spherically symmetric bubble of true vacuum.

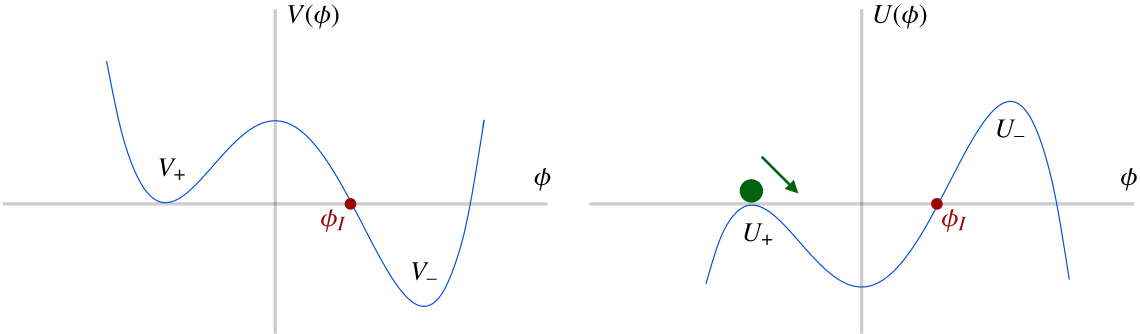

Figure 1: (Left): The potential $V(\phi)$ with its local and global minima. (Right): The inverted potential $U(\phi)$. This allows us to think in terms of the motion of a particle. Starting from ϕ_+, we then see that it can roll up to ϕ_I and bounce back. Hence the name for this type of solutions.

The probability per unit of volume Γ that such an event will occur can be determined in the semi-classical approximation $\hbar \ll 1$ and it is given by:[3]

$$\Gamma \sim e^{-B}\,(1+\mathcal{O}(\hbar))\,, \tag{2}$$

with $B = S_E(\phi_I) - S_E(\phi_-)$. The term $S_E(\phi_-)$ is the Euclidean action of the system evaluated in the false vacuum while $S_E(\phi_I)$ is the Euclidean action for the bounce solution. The bounce is the instanton solution corresponding to the nucleated bubble, i.e. solutions that extremise and give a finite value to the Euclidean version of the action that governs the dynamics of the system. Let us discuss this quantity in more detail.

2.1 The bounce

In order to easily understand the bounce, we are going to work on three specific regimes that are the implicit ones in this work:

- The first one, by analytic continuation, we will perform all our computations in the Euclidean realm ($\tau = it$).

[3]This can be deduced from the transition matrix element between the ground state $|0\rangle$ evolving in time. When $\hbar$-corrections are taken into account, the associated eigen-energy of the process is $E = \frac{1}{2}\omega\hbar + \hbar K\,e^{-B/\hbar}$, where $K \in \mathbb{C}$. Identifying the corrections with the non-Hermitian piece of the evolution operator $\mathcal{H}$ of the system, one can then relate the imaginary part of the energy E with the decay channel Γ.

- The second one will affect the shape of the potential $V(\phi)$, as we will be working in the *thin* wall approximation. This is the case when the difference between the energy density of the two vacua is really small, so that one writes $|\Delta V(\phi)| = \varepsilon$ with $|\varepsilon| \ll 1$.

- The last regime will help us to think in terms of particle dynamics. If we invert the potential, i.e. $U(\phi) = -V(\phi)$, such as shown in figure 1, we can then use a motion analogy. In this case, a particle sitting on the "lower" hill could start rolling down, up to the point of the potential where its energy will be equal to that of the starting position. As we know from high school, this is the point that the particle will reach with zero velocity and would then *bounce* back down the valley. Restoring the sign, we then see the meaning of that bouncing point: it would be the final position at which the tunneling process ends. We will call this field position as ϕ_I, which is a solution of the equation of motion and extremise the value of the Euclidean action.

Given the previous assumptions and exploiting the implicit $O(D)$ symmetry of the Euclidean space of study, so that we can rewrite all coordinates $(\tau, \bar{x})$ in terms of a "radial" coordinate $r = \sqrt{\tau^2 + x_i x^i}$, it can be shown that the Euclidean version of action (1) is

$$\begin{aligned} S_E^{\text{total}} = \int \mathrm{d}^D x \left(\tfrac{1}{2}\partial_M \phi \partial^M \phi + U(\phi)\right) =& \Omega_{D-1} \int_R^\infty \mathrm{d}r\, r^{D-1} \left[\left(\frac{\mathrm{d}\phi_+}{\mathrm{d}r}\right)^2 + U_+(\phi)\right] \\ &+ \Omega_{D-1} R^{D-1} \int \mathrm{d}r \underbrace{\left[\left(\frac{\mathrm{d}\phi_I}{\mathrm{d}r}\right)^2 + U_+(\phi_+)\right]}_{S_1} \\ &+ \Omega_{D-1} \int_0^R \mathrm{d}r\, r^{D-1} \left[\left(\frac{\mathrm{d}\phi_-}{\mathrm{d}r}\right)^2 + U_-(\phi)\right], \end{aligned} \tag{3}$$

where Ω_{D-1} is the area of a unit-radius $(D-1)$-sphere given by:

$$\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{4}$$

Note that we have divided the integration regime into three different intervals:

- That corresponding to the *outside* of the bubble (first line in Eq. (3)). The phase transition has not yet reach to this region, so the field and associated potential $U(\phi)$ correspond to the *false* configuration.

- The *instanton* solution, which corresponds to the location of the bubble (second line). We will call this piece S_1, and it represents the Euclidean action for the bounce solution.

- The *inside* volume enclosed by it (last line). The decay has already taken place here, so we write ϕ_- and $U(\phi)$.

Let us now specify our computations for a potential such that $U(\phi_+) = 0$ and $U(\phi_-) \simeq -\varepsilon$. Plugging in these considerations into the action (3), we will obtain:

$$S_E^{\text{total}} = S_E(\phi_I) - S_E(\phi_-) = \Omega_{D-1}\left(R^{D-1} S_1 - \frac{R^D}{D}\varepsilon\right), \tag{5}$$

which corresponds to the expression in the argument of Eq. (2). If we now derive the action with respect to R, we will find the radius R that extremises the value of the action. This yields:

$$B = \frac{\Omega_{D-1}}{D}\left(\frac{D-1}{\varepsilon}\right)^{D-1} S_1^D. \tag{6}$$

In this way we have found the closed-form expression for the coefficient B in the thin-wall approximation. The explicit value of this term will depend on the potential shape integrand in S_1. In any case, this value ensures that the bubble nucleation probability is maximal.

2.2 Coleman-De Lucia when gravity is present

The previous discussion was made without considering the role of gravity. When gravity is not present, any false vacuum in quantum field theory can decay. However, this is not the case when the theory is coupled to gravity. This was later studied by Coleman and de Luccia in [1], who showed that gravity can stabilise some false vacua, making them persistant.[4] Let us elaborate on this point:

The bubble, being an instanton solution, is formed at a radius R such that it minimises the Euclidean action. The formation of such a bubble has an energy cost, which is proportional to the bubble tension σ times the area of the bubble. But where does this energy come from? By conservation of energy, it must have come from reducing the vacuum energy inside its interior. This amount of energy is equal to the vacuum energy density times the volume enclosed by the bubble. This relationship between the energy contributions can be seen in Eq. (5). If the balance is *precise* so that one term compensates for the other, the bubble will be nucleated with the radius that minimises the action and remain at rest. In the case the bubble gains more

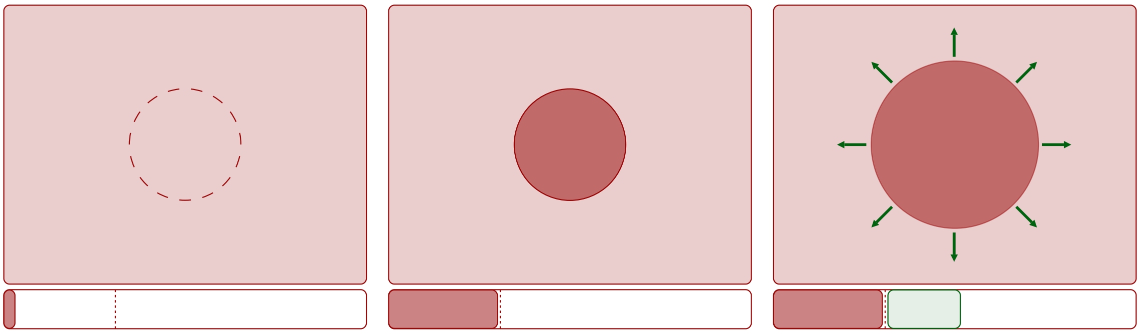

Figure 2: From left to right: Bubble nucleation requires a delicate balance between the surface energy (represented by the dashed line) and the vacuum energy difference (the bar under each rectangle). When the values are *exactly*equal to compensate each other, a bubble will nucleate at rest and remain there. However, if more energy is extracted from the decay than is required to form the bubble, the extra energy will be kinetic and used to expand the bubble.

energy than is needed to create it by reducing the vacuum energy inside it, the extra energy will provide the kinetics to expand the bubble. The bubble, which will nucleate at rest, is then accelerated outwards, asymptotically approaching the speed of light. This whole process can be understood by the expression

$$\left|V_{\text{after}} - V_{\text{before}}\right| = E_{\text{wall}} + E_{\text{kinetic}}, \tag{8}$$

where E_{kinetic} is the kinetic energy of the bubble if more energy was obtained by reducing the energy vacuum density inside. Note that when this term is zero, the tension σ, i.e. the "mass" of the bubble at rest is proportional to difference in vacuum energy discussed above.

3 Brown-Teitelboim instantons

The previous discussion only applies if the decay process occurs once. One can then wonder whether a tower of decays can be achieved within the precedent formalism. This was done by

[4] It can be shown that the bounce solution gets modified to by gravity as:

$$B_{\text{gravity}} = \frac{B}{(1+(R_0/2\Lambda_D))^2}, \tag{7}$$

where R_0 is the radius that minimises the action (5) and Λ_D is the vacuum energy of the D-dimensional spacetime, i.e. a cosmological constant.

Brown and Teitelboim in [2], where they proved that one can have a tower of vacuum decays when antisymmetric tensor fields are considered in the bubble action.

Let us now suppose a vacuum supported by a cosmological constant Λ_D which can be lowered in the presence of antisymmetric tensor field strength $F = \mathrm{d}A$, where A is the corresponding tensor field. In this case, the nucleated membrane will be charged under this field A. The Euclidean action to describing this type of instantons is given by:

$$\begin{aligned} S_E = &-\frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K \\ &+\frac{1}{(D-1)!}\int \mathrm{d}^D x\sqrt{|g|}\nabla_M\left(F^{M\cdots}A_{\ldots}\right) - \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+\frac{q}{(D-1)!}\int \mathrm{d}^{D-1}y A_{M_1\cdots M_{D-1}}\,e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}, \end{aligned} \tag{9}$$

with ϵ is the Levi-Civitta symbol. The first line represents the gravitational contribution of the whole space. In this case, the energy-momentum tensor of all matter living **on** the hypersurface (i.e. the membrane), is its tension σ. The second line ine Eq. (9) represents the presence of the field strength in the D-dimensional space and a boundary term for it, while the third line denotes the coupling of the brane to the tensor field A.

As F is a form of top degree, this means that it is proportional to the volume form:

$$F_{M_1\cdots M_D} = E(x)\epsilon_{M_1\cdots M_D}, \tag{10}$$

for some scalar function $E(x)$, i.e. the electric field. This can be shown to be constant away from the brane sources by examining the equation of motion for the tensor field A:

$$\partial_L E(x)\,\epsilon^{LM_1\cdots M_{D-1}} = -q\int \mathrm{d}^{D-1}y\,\delta\left(x^\alpha - x^\alpha(y)\right)e^{M_1}_{m_1}\cdots e^{M_{D-1}}_{m_{D-1}}\epsilon^{m_1\cdots m_{D-1}}. \tag{11}$$

This implies that the electric field $E(x)$ will jump one charge unit q across the brane.[5] This jump will be an additional contribution to lowering the vacuum energy via brane nucleation. Indeed, if one inserts Eq. (10) in the action (9) one can read off an effective cosmological constant:

$$\Lambda_{\text{eff}} = \Lambda_D + \tfrac{1}{2}\kappa_D E^2. \tag{12}$$

Thus, when the equation of motion (11) holds, we can rewrite the action (9) as:

$$\begin{aligned} S_E = &-\frac{1}{2\kappa_D}\int \mathrm{d}^D x\sqrt{|g|}\left({}^{(D)}R - 2\Lambda_D\right) + \frac{1}{2D!}\int \mathrm{d}^D x\sqrt{|g|}\,|F|^2 \\ &+\sigma\int \mathrm{d}^{D-1}y\sqrt{|h|} + \frac{1}{\kappa_D}\int \mathrm{d}^{D-1}x\sqrt{|g|}\,K, \end{aligned} \tag{13}$$

Let us now discuss the decay channel in Eq. (2) associated with the nucleation of a single bubble of *true* vacuum[6] when the tensor field A permeates the vacuum. Here we will just highlight the most important results of [2]. The curious reader interested in the detailed steps of the derivation is referred to that paper.

It can be shown that the bounce of the action (13) is given by:

$$\begin{aligned} B = \sigma\,\Omega_{D-1}R^{D-1} + \frac{1}{\kappa_D}\bigg|\bigg[&\frac{2\Lambda_i}{(D-2)}\mathrm{Vol}_D(R,\epsilon_i,\Lambda_i) \\ &+(D-1)\epsilon_i\sqrt{\frac{1}{R^2} - \frac{2\Lambda_i}{(D-2)(D-1)}}\,\Omega_{D-1}R^{D-1}\bigg]\bigg|_+^-, \end{aligned} \tag{14}$$

[5]Due to the conservation of charge, this charge difference will be carried by on the bubble's boundary, i.e. *the (mem)brane.*

[6]As the decay can occur repeatedly, we should speak of a *less false* vacuum. Let us stick to *true* to avoid confussion.

where we have dropped the "eff" superscript of the effective cosmological constant Λ and ϵ_i represents the orientation choice for the normal n_μ of the hypersurface Σ, i.e. the membrane. The area covered by a unit-radius $(D-1)$-dimensional Euclidean membrane is Ω_{D-1} and is given by:

$$\Omega_{D-1} = \int \mathrm{d}^{D-1}\xi \sqrt{^{(D-1)}g} = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{15}$$

where $\Gamma(x)$ is the usual gamma function. The volume of the inside and the complement[7] of the outside can be computed by:

$$\begin{aligned} V_D(R, \epsilon_\mathrm{i}, \Lambda_\mathrm{i}) = \int_\pm \mathrm{d}^D x \sqrt{g} &= \left(\frac{(D-1)(D-2)}{2|\Lambda_\mathrm{i}|}\right)^{D/2} \Omega_{D-1} \\ &\times \left| \int_1^{\sigma_\mathrm{i}\left[1-2\Lambda_\mathrm{i}R^2/(D-2)(D-1)\right]^{1/2}} \mathrm{d}(\cos x) \sin^{D-2} x \right|. \end{aligned} \tag{16}$$

In the case that the effective cosmological constant is $\Lambda < 0$, the trigonometric functions in Eq. (16) should be replaced by hiperbolic trigonometric functions.

In the same spirit as in the case of Coleman-De Lucia instantons, one can find the value of R that extremises the action B by deriving it with respect to R. Substituting the resulting R_0 back into B gives the extremised nucleation probability of a Brown-Teitelboim bubble in D dimensions.

References

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[7] This is the volume fraction of the background that is converted into the inside region when the bubble is created, i.e. the dashed volume in figure 2.