Special Relativity and Tensor Manipulation

Pion observation

The half-life of the elementary particle called the pion is $2.5 \cdot 10^{-8}$ s when the pion is at rest with respect to the observer measuring its decay time. Show that pions moving at speed $v = 0.999c$ has a half-life of $5.6 \cdot 10^{-7}$ s, as measured by an observer at rest.

Car-Garage Paradox

Consider a couple, Mark and Mina, that just bought a car of really small height and $20ft$ length. When they go home they realize that their garage is only 10ft long. In order to avoid the problem Mina says that she can drive the car in the garage with speed $u = 0.866c$ (which means that $\gamma = 2$) so in the system of reference of the garage the car will be smaller and it will exactly fit. However Mark disagrees, because in the system of reference of the car, the garage will be smaller and the car will not fit at all.

1. Justify the two different opinions.
2. The door mechanism works as follows, when the car hits the wall of the garage, the door closes automatically. What will happen if they decide to go through with the experiment? Will the car be inside the garage when the door closes or not?

Three Observers

Three events, A, B, C are seen by observer $\mathcal{O}$ to occur in the order ABC. Another observer, $\mathcal{O}'$, sees the events to occur in the order CBA. Is it possible that a third observer sees the events in the order ACB? Support your conclusion by drawing a spacetime diagram.

Tensor Game

This problem is a simple game. Identify which of the following equations could be valid tensor equations; for the ones that cannot be, say why not. Here I mean tensors e.g. under the Lorentz group (or maybe some more general transformations) where we must distinguish between co-variant (lower) and contravariant (upper) indices.

(a) $R^a_{man} = T_{mn}$

(b) $\text{☺}_a \omega_{bc} = \text{♃}_{ab}$

(c) $\lozenge_{a\aleph\aleph} = \text{Ϭ}_a$

(d) $\text{♫}_{ab} + \text{☼}_{ac} = \Upsilon_{bc}$

(e) $\natural_a(\sharp_b + \flat_b) = \text{☻}_{ab}$

(f) $\text{☹}_a \bigstar_b = \text{☽}_{ab}$

Tensor Manipulation

1. Let $A_{\mu\nu}$ be a $(0,2)$-tensor and B^{μ} a $(1,0)$-tensor (a vector). Show that $A_{\mu\nu}B^{\nu}$ is a covector/dual vector/one-form (i.e. transforms like a co-vector) and that $A_{\mu\nu}B^{\mu}B^{\nu}$ is a scalar.
2. Given the components of a $(2,0)$ tensor $M^{\alpha\beta}$ as the matrix

$$M^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix}.$$

   find:

   (a) the components of the symmetric tensor $M^{(\alpha\beta)}$ and the antisymmetric tensor $M^{[\alpha\beta]}$;
   (b) the components of M^{α}_{β};
   (c) the components of $M_{\alpha}{}^{\beta}$;
   (d) the components of $M_{\alpha\beta}$.
   (e) For the $(1,1)$ tensor whose components are M^{α}_{β}, does it make sense to speak of its symmetric and antisymmetric parts? If so, define them. If not, say why.
   (f) Raise an index of the metric tensor to prove $\eta^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$.

Tensor Manipulation 2

For any 2 -tensor $T_{\mu\nu}$ (in four dimensions) we define its symmetric and antisymmetric part respectively,

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right), \qquad T_{[\mu\nu]} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right).$$

1. Is it true that for any tensor $T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}$? How many independent components do $T_{(\mu\nu)}$ and $T_{[\mu\nu]}$ have?
2. If $S_{\mu\nu}$ and $A_{\mu\nu}$ are purely symmetric and antisymmetric 2 -tensors respectively, prove that for a generic $T_{\mu\nu}$

$$T_{\mu\nu}S^{\mu\nu} = T_{(\mu\nu)}S^{\mu\nu}, \qquad T_{\mu\nu}A^{\mu\nu} = T_{[\mu\nu]}A^{\mu\nu}.$$

3. Now consider the case of an arbitrary 3 -tensor $T_{\mu\nu\rho}$. How many independent components does it have? Write explicitly the form of $T_{(\mu\nu\rho)}$ and $T_{[\mu\nu\rho]}$.
4. For an arbitrary 3-tensor $T_{\mu\nu\rho}$ is it true that $T_{\mu\nu\rho} = T_{(\mu\nu\rho)} + T_{[\mu\nu\rho]}$?
5. Suppose A is an antisymmetric $(2,0)$ tensor, B a symmetric $(0,2)$ tensor. Prove: $A^{\alpha\beta}B_{\alpha\beta} = 0$.

Tensor Manipulation 4

Imagine we have a tensor $X^{\mu\nu}$ and a vector V^{μ}, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad V^{\mu} = (-1, 2, 0, -2).$$

Find the components of: $X^{\mu}_{\nu}, X_{\mu}{}^{\nu}, X^{(\mu\nu)}, X_{[\mu\nu]}, X^{\lambda}{}_{\lambda}, V^{\mu}V_{\mu}, V_{\mu}X^{\mu\nu}, V_{\mu}X^{[\mu\nu]}V_{\nu}$.

Future sight

The energy- momentum tensor, $T^{\mu\nu}$, is a symmetric $(2,0)$ tensor. For something called a "perfect fluid" it has the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{1}$$

in the rest frame of the fluid. Note that ρ is the energy density and p is the pressure.

1. Suppose that the fluid is at rest in the inertial frame S. Find $T'^{\alpha\beta}$ in the frame S', where S' is moving with velocity u along the x-direction.
2. Find a relation $p = p(\rho)$ such that $T^{\mu\nu}$ is the same in any inertial frame.

Challenge Problem (Carroll 1.11.2)

Imagine that space (not spacetime) is actually a finite box, or in more sophisticated terms, a three-torus, of size L. By this we mean that there is a coordinate system $x^\mu = (t, x, y, z)$ such that every point with coordinates (t, x, v, z) is identified with every point with coordinates $(t, x + L, y, z)$, $(t, x, y + L, z)$, and $(t, x, y, z + L)$. Note that the time coordinate is the same. Now consider two observers; observer A is at rest in this coordinate system (constant spatial coordinates), while observer B moves in the x-direction with constant velocity v. A and B begin at the same event, and while A remains still, B moves once around the universe and comes back to intersect the worldline of A without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by A and B? Is this consistent with your understanding of Lorentz invariance?

SPECIAL RELATIVITY AND REVIEW

N11 PION OBSERVATION

WELL, WELL! THIS IS A SIMPLE ONE. RECALL THAT "A LITTLE MANEUVER IS GONNA COST US 51 YEARS" (INTERSTELLAR).

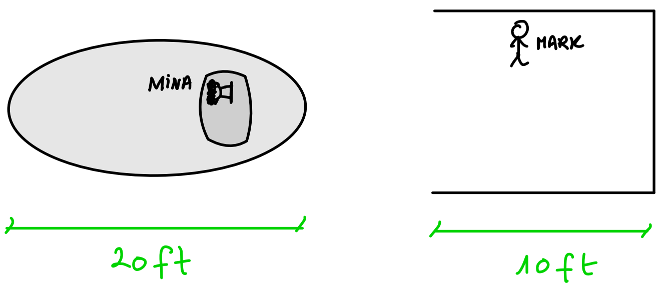
AN EXTERNAL OBSERVER WILL MEASURE MORE TIME AT REST THAN AN OBSERVER MOVING QUITE FAST / IN A HIGHLY CURVED SPACE.

TIME DILATION IS GIVEN BY $t' = \gamma t$, $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$;

SO $\tau_{OBS} = \gamma \, \tau_{PION} \cong 5{,}6 \cdot 10^{-7}\,s$.

N12 CAR - GARAGE PARADOX

WE CAN AGREE THAT THE PROBLEMS OF THIS COUPLE WOULD NOT BE SO IF THEY HAD BOUGHT A BIKE. SO WE HAVE:



THE SPEED OF THE CAR IS $v = 0.866c \Rightarrow \gamma = 2$.

FROM THE CAR PERSPECTIVE, IT IS AT REST, SO $L_{car} = 20ft$.

BUT THE GARAGE MOVES TOWARDS IT AT $v = -0.866c$, SO SPACE CONTRACTION!

ⓐ

$$L_{GARAGE} = \frac{L_{G\,REST}}{\gamma} = 5ft.$$

MINA WILL LEAVE HER FLESH ON THE WALL.

(AND KILL MARK ALSO)

FROM MARK PERSPECTIVE, THE GARAGE IS AT REST, SO ITS LENGTH REMAINS UNTOUCHED. THE CAR'S LENGTH WOULD ALSO CONTRACT, AS:

$$L_{CAR} = \frac{L_{CAR\ REST}}{\gamma} = 10\,ft$$

SO BOTH ARE RIGHT. FROM THE POINT OF VIEW OF A NEIGHBOUR, JUST TWO IDIOTS DISCUSING IMPOSIBLE THINGS.

(b) NOTICE THAT SPACETIME EVENTS ARE THE SAME IN ALL FRAMES!

ONE SHOULD CAREFULLY COMPUTE THIS. LETS USE 4D COORDINATES. FROM MARK'S PERPECTIVE WE HAVE:

- THE FRONT PART ENTERS $(t = 0,\ x = 0)$
- MARK WAVES TO MINA $(t = t_w,\ x_w = 5)$
- THE BACK PART ENTERS $(t = t_b,\ x_b = 0)$
- THE FRONT PART HITS WALL $(t = t_h = t_b,\ x_h = 10)$
- DOOR CLOSES $(t_c = t_h + \delta,\ x = 0)$

WHAT IS δ? WELL, THE TIME IT WILL TAKE THE SIGNAL TO GO THROUGH THE CIRCUIT AND CREATE A DOOR OF LIGHT.

THE SPECIAL RELATIVITY EQS ARE:

$$t' = \gamma(t - \beta x)$$
$$x' = \gamma(x - \beta t)$$

SO, FROM CAR PERSPECTIVE, WE JUST APPLY THIS TRANSFORMATIONS.

MINA'S PERSPECTIVE:

- THE FRONT PART ENTERS $(t' = 0,\ x' = 0)$
- MARK WAVES TO MINA $(t_w' = \gamma(t_w - S\beta),\ x_w' = \gamma(x_w - \beta t_w))$
- THE BACK PART ENTERS $(t_b' = \gamma t_b,\ x_b' = -\gamma\beta t_L)$
- THE FRONT PART HITS WALL $(t_h' = \gamma(t_h - \beta x_h),\ x_h' = \gamma(x_h - \beta t_h))$
- DOOR CLOSES $(t_c' = \gamma(t_c),\ x_c' = -\gamma\beta t_c)$

OBSERVE THAT IF WE COMPARE t_h' AND t_b' WE SEE THAT:

$$\gamma(t_b - \beta x_u) < \gamma t_b \quad , \text{ AS } \quad \gamma\beta x_u > 0 .$$

SO FRONTS HITS WALL BEFORE TRUNK GOES IN. NOW, WILL THE "LASER" DOOR TURN ON BEFORE THE CAR IS 'IN'? WELL.

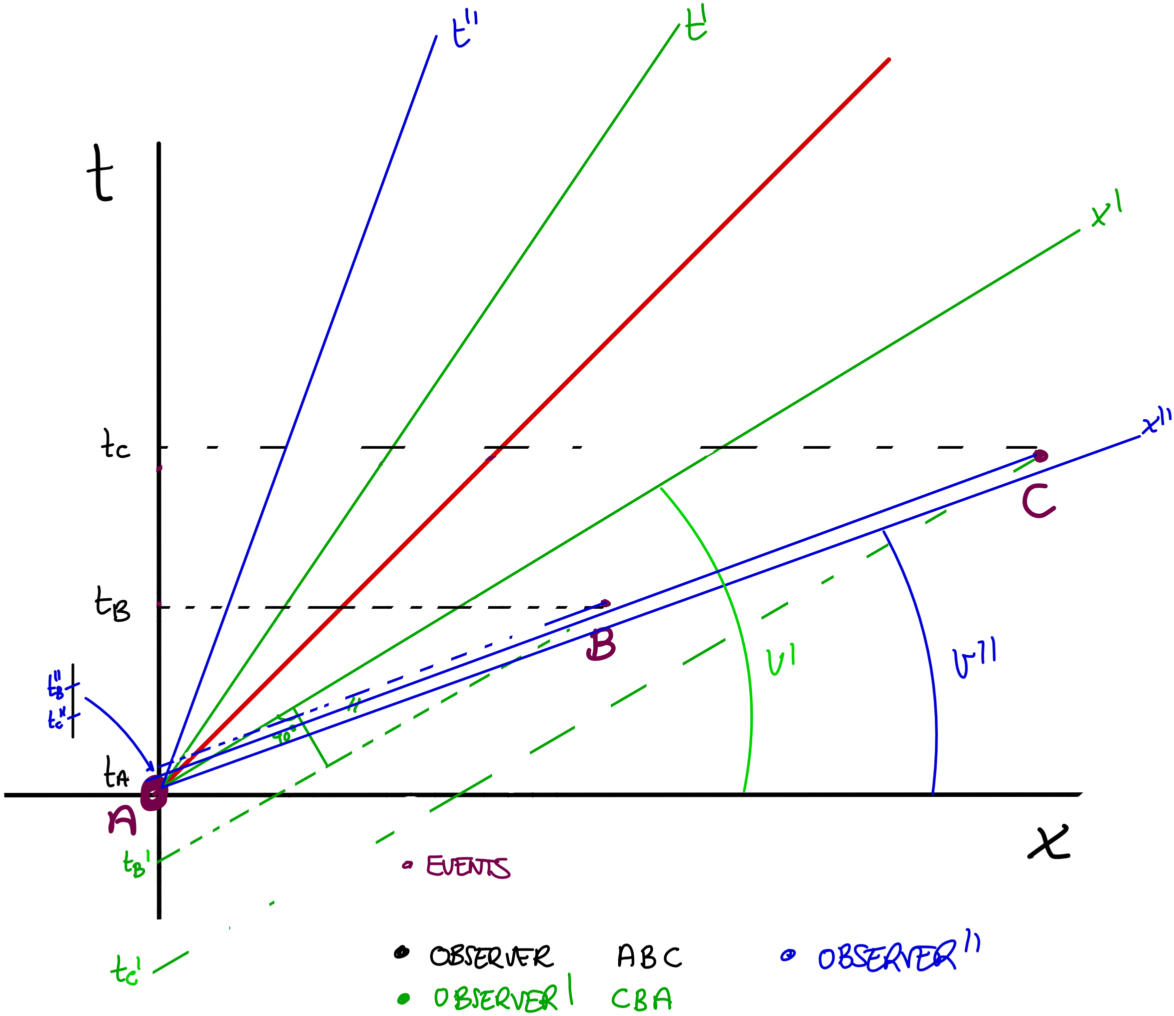
$$\gamma(t_b - \beta x_u) < \gamma(t_b + \delta) .$$

SO THE CAR GOES IN BEFORE THE DOOR POPS UP.

N13 THREE OBSERVERS

IMPORTANT THINGS TO KNOW:

- THE FASTER YOU MOVE, THE "SMALLER" YOU LIGHT CONE IS:
- THIS AFFECTS WHEN YOU SEE SOME EVENTS.



OBSERVE $0 < v'' < v'$, SO WE HAVE THIS EVENT DISTRIBUTION.

THE EVENTS ARE DETERMINED BY THE INTERSECTION OF t-AXIS AND THE x-AXIS THAT POPS UP FROM THE EVENT.

N14 TENSOR GAME

RECALL THAT A (p,q) TENSOR ARE OBJECTS WITH p CONTRAVARIANT INDICES AND q COVARIANT ONES THAT TRANSFORM NICELY UNDER LORENTZ TRANSFORMATIONS.

a) YES! $R^a{}_a = R$ IS CONTRACTION OF 2 SAME INDICES.

b) NO! WHERE IS c?

c) NOP. MISSING N INDICES.

d) ALMOST... BUT NO. (NO MATCHING INDICES)

e) COULD BE, AS $\natural_a \sharp_b \sim \frac{1}{2} ☺_{ab}$

f) ALSO POSSIBLE.

N15 TENSOR MANIPULATION.

a) IN THE PREVIOUS EXERCISE I MENTIONED THAT A TENSOR TRANSFORMS NICELY UNDER LORENTZ, AS:

$$T^{\mu\nu\rho\cdots}{}_{\alpha\beta\gamma\cdots} = \frac{\partial y^\rho}{\partial x^\psi}\frac{\partial y^\nu}{\partial x^\theta}\frac{\partial y^\mu}{\partial x^\delta}\frac{\partial x^\xi}{\partial y^\gamma}\frac{\partial x^\phi}{\partial y^\beta}\frac{\partial x^\omega}{\partial y^\alpha}T^{\delta\theta\psi}{}_{\omega\phi\xi}$$

SO A ONE FORM $(A_\mu) \Rightarrow \quad A_\alpha = \dfrac{\partial x^\beta}{\partial y^\alpha}A_\beta$

WHILE A VECTOR $(v^\mu) \Rightarrow \quad v^\alpha = \dfrac{\partial x^\alpha}{\partial y^\beta}v^\beta$

THEN

$$A_{\mu\nu}B^\nu = \underbrace{\frac{\partial x^\alpha}{\partial y^\mu}}_{\text{ONE AS 1-FORM.}}\cancel{\frac{\partial x^\beta}{\partial y^\nu}}\,\cancel{\frac{\partial y^\nu}{\partial x^\beta}}\,A_{\alpha\beta}B^\beta \Rightarrow A_{\mu\beta}B^\beta = A_{\mu\nu}B^\nu$$

DUMMY

$$A_{\mu\nu}B^\mu B^\nu = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta}A_{\alpha\beta}B^\alpha B^\beta \longrightarrow \text{AS SCALAR}$$

(b) $M^{(\alpha\beta)} = \frac{1}{2}\left(M^{\alpha\beta} + M^{\beta\alpha}\right)$ SYMM

$M^{[\alpha\beta]} = \frac{1}{2}\left(M^{\alpha\beta} - M^{\beta\alpha}\right)$ ANTISYM (NO DIAG)

$M^{(\alpha\beta)} + M^{[\alpha\beta]} = M^{\alpha\beta}$.

ALSO $M^{\beta\alpha} = (M^{\alpha\beta})^{T}$

$$M^{\alpha\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \qquad M^{\beta\alpha} = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

$$M^{(\alpha\beta)} = \begin{bmatrix} 0 & 1 & 1 & 1/2 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & -1/2 \\ 1/2 & 1 & -1/2 & 0 \end{bmatrix} \qquad M^{[\alpha\beta]} = \begin{bmatrix} 0 & 0 & -1 & -1/2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3/2 \\ 1/2 & -1 & -3/2 & 0 \end{bmatrix}$$

TO RAISE INDEX OR LOW, USE METRIC. IN GR IS NOT $\eta_{\mu\nu}$, BUT COULD BE SOME TIMES...

SO; $M^{\alpha}{}_{\beta} = \eta_{\beta\gamma} M^{\alpha\gamma}$

DUMMY

So:

$$M^0{}_0 = \eta_{00} M^{00} + \cancel{\eta_{01}} M^{01} + \cancel{\eta_{02}} M^{02} + \cancel{\eta_{03}} M^{03}$$

η is DIAGONAL

$$M^0{}_1 = \cancel{\eta_{10}} M^{00} + \eta_{11} M^{01} + \cancel{\eta_{12}} M^{02} + \cancel{\eta_{13}} M^{03}$$

$$\vdots$$

ONE FINDS:

$$M^\alpha{}_\beta = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}$$

SAME TO FIND $M_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta} M^{\gamma\delta} = \eta_{\alpha\gamma} M^\gamma{}_\beta$

so $$M_{\alpha\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & 2 \\ -2 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix}$$

AND $$M_\alpha{}^\beta = \eta_{\alpha\gamma} M^{\gamma\beta} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 2 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

IN THE CASE WE WANT TO TALK ABOUT (ANTI) SYM FOR $M^{\alpha}{}_{\beta}$.

THE DEFINITION WORKS ONLY FOR 2 INDEX DOWN OR 2 UP. BUT!

WE CAN FIND (ANTI) SYM PARTS IF WE LOW ONE INDEX.

$$M^{(\alpha\beta)} = \frac{1}{2}\left(M^{\alpha\beta} + M^{\beta\alpha}\right)$$

$$\eta_{\alpha\gamma} M^{(\gamma\beta)} = \frac{1}{2}\eta_{\alpha\gamma}\left(M^{\gamma\beta} + M^{\beta\gamma}\right)$$
$$= \frac{1}{2}\left(M_{\alpha}{}^{\beta} + M^{\beta}{}_{\alpha}\right)$$

LASTLY, PROVE $\eta^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}$.

$$\eta_{\alpha\gamma}\eta^{\gamma\beta} = \eta_{\alpha}{}^{\gamma}\eta_{\gamma}{}^{\beta} = \left(\eta^{-1}\eta\right)_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}$$

DUMMY INDICES CAN BE 'RAISE/LOWN' IN PAIRS.

N22 TENSOR MANIPULATION 2

RECALL THAT SYM PART $T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$

ANTI PART $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$

(a) YES! $\exists$ THAT ARE FULLY ANTISYM. OTHER FULL SYM. AND MIXTURES. IN FACT,

$$T_{[\mu\nu]} + T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + \cancel{T_{\nu\mu}} + T_{\mu\nu} - \cancel{T_{\nu\mu}}) = T_{\mu\nu}$$

OJO. OJITO. OJAZO. THIS IS NOT TRUE FOR $p = q > 2$ (a.k.a Rank > 2) AS THERE MAY $\exists$ EXTRA SYMS.

REGARDING INDEPENT COMPONENTS, THE SYM HAS DIMENSION:

$$\mathrm{Dim}\ \mathrm{Sym}^k = \binom{D+k-1}{k} = \frac{(D+k-1)!}{k!\,(D+k-1-k)!} = \frac{(D+k-1)!}{k!\,(D-1)!};$$

WHERE D = DIM SPACE TIME AND k = RANK.

$$\text{SO}\quad \mathrm{Dim}\ \mathrm{Sym} = \frac{(4+2-1)!}{2!\,3!} = \frac{5!}{2!} = \frac{5.4.3}{3.2} = 10$$

WHILE, FOR THE ANTISYM PART $\Rightarrow$ 6 ; (BECAUSE DIM $T = D^K$)

IN GENERAL, FOR RANK = 2 TENSORS ;

$$\mathrm{DIM}\ T_{(\mu\nu)} = \frac{(D+1)!}{2(D-1)!} = \frac{(D+1)D}{2}$$

$$\mathrm{DIM}\ T_{[\mu\nu]} = D^K - \mathrm{DIM}\ T_{(\mu\nu)} = \frac{2D^2 - D^2 - D}{2} = \frac{D(D-1)}{2}$$

Ⓑ $S_{\mu\nu}$ = SYM $\quad S_{\mu\nu} = S_{\nu\mu}$ *

$A_{\mu\nu}$ = ANTI $\quad A_{\mu\nu} = -A_{\nu\mu}$ **

- $$T_{(\mu\nu)} S^{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right) S^{\mu\nu} = \frac{1}{2} T_{\mu\nu} S^{\mu\nu} + \frac{1}{2} T_{\nu\mu} S^{\mu\nu} \overset{*}{=}$$
$$\overset{*}{=} \frac{1}{2} T_{\mu\nu} S^{\mu\nu} + \frac{1}{2} T_{\nu\mu} S^{\nu\mu} = T_{\mu\nu} S^{\mu\nu}.$$

- $$T_{[\mu\nu]} A^{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) A^{\mu\nu} = \frac{1}{2} T_{\mu\nu} A^{\mu\nu} - \frac{1}{2} T_{\nu\mu} A^{\mu\nu} \overset{**}{=}$$
$$\overset{**}{=} \frac{1}{2} T_{\mu\nu} A^{\mu\nu} + \frac{1}{2} T_{\nu\mu} A^{\nu\mu} = T_{\mu\nu} A^{\mu\nu}.$$

Ⓒ DIM OF A TENSOR IS D^K, SO $T_{\mu\nu\rho}$ IS D^K. IN 4D IS 64 ENTRIES. (IN SUGRA YOU HAVE OBJECTS AS $G_{\mu\nu\rho\sigma\delta\alpha\beta}$ IN 10D)

$$T_{(\mu\nu\rho)} = \frac{1}{k!}\left(T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu} + T_{\nu\mu\rho} + T_{\rho\nu\mu} + T_{\mu\rho\nu}\right);$$

$$T_{[\mu\nu\rho]} = \frac{1}{k!}\Big(\underbrace{T_{\mu\nu\rho} + T_{\nu\rho\mu} + T_{\rho\mu\nu}}_{+\ \text{PERMUTATION}} \underbrace{- T_{\nu\mu\rho} - T_{\rho\nu\mu} - T_{\mu\rho\nu}}_{-\ \text{PERMUTATION}}\Big)$$

(d) BUT $T_{(\mu\nu\rho)} + T_{[\ ]} \neq T_{\mu\nu\rho}$!

(e) $A^{\alpha\beta} B_{\alpha\beta} = -A^{\beta\alpha} B_{\beta\alpha}$ = SCALAR, SUCH THAT EQUAL TO ITSELF = 0.

SO, ANTISYM CONTRACT WITH SYM $= 0$!

022 TENSOR MANIPULATION 4

$$X^{\mu\nu} = \begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix} \quad \text{AND} \quad V^{\mu} = \begin{pmatrix} -1 & 2 & 0 & -2 \end{pmatrix}$$

TENSOR MANIPULATION REQUIRES THE METRIC! SO FAR WE WORK IN FLAT SPACE AS:

$$\eta_{\mu\nu} = \text{Diag}\{-1\ 1\ 1\ 1\}$$

DUMMY = AVOID REP

- $X^{\mu}{}_{\nu} = \eta_{\nu\alpha} X^{\mu\alpha}$; AS η IS DIAG $\Rightarrow \nu = \alpha$.

$$X^{0}{}_{0} = \eta_{00} X^{00} = -2$$

$$X^{0}{}_{1} = \eta_{11} X^{01} = 0$$

$$\vdots$$

$$X^{\mu}{}_{\nu} = \begin{bmatrix} -2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & -2 \end{bmatrix}$$

- $x_\mu{}^\nu = \eta_{\mu\alpha} x^{\alpha\nu} = \begin{bmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{bmatrix}$

- $X^\lambda{}_\lambda = \mathrm{Trace}\,[x] = -2+0+0+(-2) = -4$

- $x^{(\mu\nu)} = \frac{1}{2}\left(x^{\mu\nu} + x^{\nu\mu} \right) = \frac{1}{2}\left(X + X^T \right)$

$$= \begin{bmatrix} 2 & -1/2 & 0 & -3/2 \\ -1/2 & 0 & 2 & 3/2 \\ 0 & 2 & 0 & 1/2 \\ -3/2 & 3/2 & 1/2 & -2 \end{bmatrix}$$

- $x_{[\mu\nu]} = \frac{1}{2}\left(X_{\mu\nu} - X_{\nu\mu} \right) = \begin{bmatrix} 0 & -1/2 & -1 & -1/2 \\ 1/2 & 0 & 1 & 1/2 \\ 1 & -1 & 0 & -1/2 \\ 1/2 & -1/2 & 1/2 & 0 \end{bmatrix}$

- $V^\mu V_\mu$ = WELL KNOW SCALAR PRODUCT! BUT IN A NON-EUCLIDEAN SPACE.

$$= \eta_{\mu\nu} v^\mu v^\nu = v^T \eta v = 7$$

- $V_\mu X^{\mu\nu} = \eta_{\mu\alpha} V^\alpha X^{\mu\nu} = \{-4, 2, 9, -1\}$
- $V_\mu X^{[\mu\nu]} V_\nu = \eta_{\mu\alpha} V^\alpha X^{[\mu\nu]} \eta_{\nu\beta} X^\beta = 0$!

023 FUTURE SIGHT

WE WILL SEE WHAT THIS OBJECT STANDS FOR, LATER IN THIS COURSE. LET'S THINK OF IT AS A (2,0)-TENSOR.

$$T^{\mu\nu} = \text{Diag}\{\rho, p, p, p\} \leftarrow \text{symmetric.}$$

@ THIS IS JUST ABOUT PERFORMING A BOOST (AS IN ELECTRODYNAMICS!)

BOOSTS CAN BE PERFORMED BY CONTRACTION W/ LORENTZ MATRICES AS:

$$\Lambda^{\mu}{}_{\nu} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ASSUME $c = 1 \rightarrow \beta = u$

SO: $T'^{\alpha\beta} = \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu} T^{\mu\nu}$;

WE JUST NEED TO COMPUTE PIECE BY PIECE:

$$T'^{00} = \Lambda^0{}_\mu \Lambda^0{}_\nu T^{\mu\nu} = \Lambda^0{}_0 \Lambda^0{}_0 T^{00} + \Lambda^0{}_1 \Lambda^0{}_1 T^{11} = \gamma^2(\rho + u^2 p)$$

$$T'^{0i} = T'^{i0} = \Lambda^0{}_\mu \Lambda^i{}_\nu T^{\mu\nu} \rightarrow$$

- iff $i = 1 \Rightarrow T'^{01} = -\gamma^2 u(\rho + p)$

  Else $\Rightarrow T'^{0i} = T^{0i}$.

$$T'^{11} = \Lambda^1{}_\mu \Lambda^1{}_\nu T^{\mu\nu} = (\Lambda^1{}_0)^2 T^{00} + (\Lambda^1{}_1)^2 T^{11} = \gamma^2(u^2\rho + p)$$

$$T'^{ij} = T^{ij} \qquad \forall i,j \neq 0,1.$$

So
$$T'^{\mu\nu} = \begin{bmatrix} \gamma^2(\rho + u^2 p) & -\gamma^2 u(p+\rho) & 0 & 0 \\ -\gamma^2 u(\rho+p) & \gamma^2(u^2\rho + p) & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

(b) IF $T''^{\mu\nu} = T^{\mu\nu} \Rightarrow$

$$T'^{0\mu} = T^{0\mu} \Rightarrow -\gamma^2 u(\rho + p) = 0 \,!$$

So $\boxed{p = -\rho}$ IF THIS IS TRUE:

$$T'^{00} = T^{00} \Rightarrow \gamma^2(\rho^2 - u^2\rho) = \rho \checkmark \quad \left(\text{B.C } \gamma^2 = \frac{1}{1-u^2}\right)$$

$$T'^{11} = T^{11}.$$

THE SIMPLY FACT THAT $p = -\rho$ CORRESPONDS TO A SPECIFIC CASE OF THE EQUATION OF STATE IN COSMOLOGY (SPOILER !)

IT GOES AS :

$$p = w\rho \ , \quad w/ \quad w \in \mathbb{R}.$$

IF $w = -1 \Rightarrow$ WE TALK ABOUT DARK ENERGY (a.k.a $\Lambda g_{\mu\nu}$ IN EINSTEIN EQ)

Metrics and Physics

Gravitational Time Dilation

Bob is at home on the surface of the Earth (radius R_E) and Alice is in a circular orbit of radius R. Assume that the gravitational field is weak and can be approximated by the following line element:

$$ds^2 = -(1+2\phi(x))dt^2 + (1-2\phi(x))dx^2,$$

where $\phi(x)$ is the Newtonian potential. Given a time interval Δt compute the elapsed proper time for both Bob and Alice, and show that they are equal for $R = \frac{3}{2}R_E$.

Rocket Passanger

Study a rocket passenger who feels "gravity" because he is being accelerated in flat spacetime.

1. Describe the 4 -velocity and 4 -acceleration.
2. Using the above describe his motion relative to an inertial reference frame. Consider, for simplicity, an observer who feels always a constant acceleration g and everything lives in two dimensions.

Some coordinate transformations

Consider $\mathbb{R}^3$ as a manifold with the flat Euclidean metric and coordinates (x, y, z). Introduce cylindrical coordinates (r, θ, z), related to (x, y, z) by

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

1. find the coordinate transformation matrix $\partial x^\mu/\partial x^{\mu'}$ between (x, y, z) and (r, θ, z)
2. find the expression for the line element ds^2 in the new coordinate system
3. a particle moves along a parameterized curve given by

   $$x(\lambda) = \cos\lambda, \quad y(\lambda) = \sin\lambda, \quad z(\lambda) = \lambda$$

   Express the path of the curve in the (r, θ, z) system.
4. calculate the components of the tangent vector to the curve in both the Cartesian and cylindrical coordinate systems.
5. consider the vectors fields $V = x\partial_y - y\partial_x + \partial_z$ and $W = \partial_y$. Compute their Lie bracket commutator.

Tensor coordinate transformation

Consider a $(0,2)$ tensor on a two-dimensional manifold, whose components in a coordinate system (x, y) read

$$S_{\mu\nu} = \begin{pmatrix} 1 & a \\ 0 & x^2 \end{pmatrix}.$$

Now consider new coordinates

$$x' = \frac{2x}{y}, \quad y' = \frac{y}{2}.$$

What are the components of S in the new coordinate system?

Consider a general vector A^μ. Does $\partial_\nu A^\mu$ transform like a $(1,1)$-tensor?

Free particle action

Consider the action of a free particle

$$S = \int d\lambda \left(-m\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}\right), \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}.$$

Suppose we reparametrize the worldline according to $\lambda \to s(\lambda)$. Use the chain rule to show that this change of parametrization preserves S. Find the equations of motion by varying the action.

Geodesic equation

Compute the timelike geodesics for the following metric:

$$ds^2 = t^{-2}\left(-dt^2 + dx^2\right).$$

(Hint: use the symmetries of the Lagrangian and recall we only need to work out $x(t)$.)

Free particle on the sphere

Imagine a particle with mass m that is forced to move on a 2-dimensional sphere of radius R. The particle is moving on the sphere with no additional forces acting on it. (The problem can be a simplified model of the physical system of a pendulum on a ball joint.)

1. Write down the Lagrangian of the system, the metric and the infinitesimal line element.
2. Find the Christoffel symbols and the equation of motion for the particle.

Geodesic on a disk

The spatial part of the metric of a rotating disk is given by

$$ds^2 = dr^2 + \frac{r^2}{1 - r^2\omega^2} d\theta^2.$$

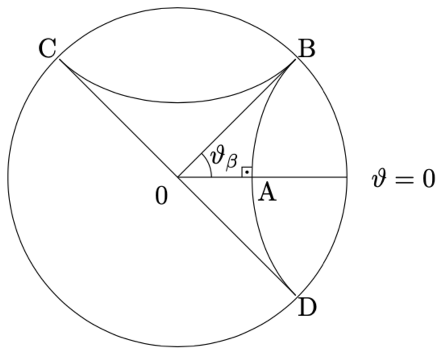


Figure 1: A geodesic on a rotating disk.

1. Write down the geodesic equations.
2. Consider the second-order equation for $\theta(s)$. Show that the integral of motion of the equation is

$$\frac{d\theta}{ds} = \frac{\alpha}{r^2}\left(1 - r^2\omega^2\right),$$

where α is a constant of integration. Assuming $L = 1$, show that

$$\frac{dr}{ds} = \pm\sqrt{\beta - \frac{\alpha^2}{r^2}},$$

where β is a constant to be determined. Finally conclude that

$$\frac{dr}{d\theta} = \pm\frac{r^2\sqrt{1 + \alpha^2\omega^2 - \frac{\alpha^2}{r^2}}}{\alpha\left(1 - r^2\omega^2\right)}.$$

By integrating this expression we can in principle solve for $r(\theta)$.
3. Consider a geodesic passing through $(r_0, 0)$ and having $\frac{dr}{ds} = 0$. Express α in terms of r_0.
4. Find the geodesic corresponding to $\alpha = 0$.
5. Show that the geodesics always cross the boundary $r_\star = \frac{1}{\omega}$ at a right angle ($\mathrm{A} = (r_\star, 0)$).
6. Find the angle ϕ between two geodesics which go through the same point, expressed in terms of $\alpha_1, \alpha_2, \omega$ and the r-coordinate at the point where they meet.

Expanding Universe

The metric for an expanding universe (the so-called Friedman-Robertson-Walker metric) is given by

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j.$$

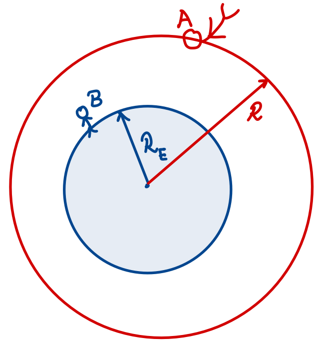
1. supposing that $a(t)$ is given, find the geodesics.
2. find the solution for a null geodesic, assuming for simplicity that $y, z = \text{const}$.

Challenge Problem

Consider a charged particle sitting on the surface of the Earth. According to the equivalence principle, it should behave in the same way as a particle accelerating in outer space. Does this mean it emits radiation? (Hint: this is a famous problem known as the 'Paradox of a charge in a gravitational field'. You are encouraged to read up on it in the literature!)

METRICS AND PHYSICS

N21 GRAVITATIONAL TIME DILATION



WE ARE GIVEN A LINE INVARIANT.

$$ds^2 = -f(x)\,dt^2 + g(x)\,dx^2$$

WITH $f(x) = 1 + 2\phi(x)$

$g(x) = 1 - 2\phi(x)$

WE FACE PLENTY OF THINGS IN THIS PROBLEM. WE SHOULD FIRST CLARIFY WHAT WEAK APPROXIMATION IS, FROM A MATHEMATICAL PERSPECTIVE.

RECALL THAT $ds^2 = -d\tau^2$ IN SR AND GR. THE ONLY THING THAT CHANGES IS THE METRIC, AS WE NOW HAVE SOME CURVATURE SO:

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu.$$

IN A WEAK FIELD APPROX, THIS MEANS THAT WE ARE

ALMOST IN A FLAT SPACE ($\eta_{\mu\nu}$) WITH CORRECTIONS ($h_{\mu\nu}$) s.t.:

$$g_{\mu\nu} \cong \eta_{\mu\nu} + \xi h_{\mu\nu}, \quad \xi \in \mathbb{R}, \quad |\xi| \ll 1.$$

TO COMPUTE THE PROPER TIME OF BOTH THEM, WE CAN INTEGRATE THEM.

$$d\tau = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} ;$$

$$\Delta\tau = \int \sqrt{\ \cdots\ } \Rightarrow \text{EXTRACT SOME AFFINE PARAMETER}$$

$$\Delta\tau = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda^2} = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

SO $\frac{dx^\mu}{d\lambda} = \dot{x}^\mu$ = 4-VELOCITY IN $\eta_{\mu\nu}$. THEN:

$$\Delta\tau = \int \sqrt{-(\underbrace{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}_{u^\mu u_\mu = -1} + \xi h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \ {}^{*})}\ d\lambda \Rightarrow$$

$$* \ \xi(h_{00}\dot{x}^0\dot{x}^0 + \underbrace{h_{ii}\dot{x}^i\dot{x}^i}_{\dot{x}^i\dot{x}_i = \bar{v}^2}) = \xi(-h_{00} + \bar{v}^2)$$

$$\Delta\tau = \int \sqrt{-(-1 + \xi(-h_{00} + \bar{v}^2))}\, d\lambda \Rightarrow \text{EXPAND IN SMALL } \xi,$$

a.k.a. $\sqrt{1-x} = 1 + \frac{1}{2}x \ldots \Rightarrow$

$$\Delta\tau \cong \int dt \left(1 - \frac{1}{2}\left(-h_{00} + \bar{v}^2\right)\right) =$$

$$\cong \int dt \left(1 + \frac{1}{2} h_{00} - \frac{1}{2}\bar{v}^2 \right)$$

SO THIS HOW THE PROPER TIME RELATES TO THE OVERALL "t" TIME.

THEN, IT IS EASY TO SEE THAT:

$$ds^2 = -(\underbrace{1}_{\eta_{00}} + \underbrace{2\phi(x)}_{h_{00}}) dt^2$$

HENCE:

$$\Delta\tau \cong \int dt \left(1 + \frac{1}{2} 2\phi - \frac{1}{2}\bar{v}^2 \right);$$

AND $\phi = -\frac{GM}{R}$, SO NOW:

$$\boxed{\Delta\tau = \int dt \left(1 - \frac{GM}{R} - \frac{1}{2}\bar{v}^2 \right)}$$

WITH EXPRESSION IN OUR POWER, WE CAN COMPUTE BOTH CASES AS:

$$\Delta\tau_R = \int dt \left(1 - \frac{GM}{R_E} - 0 \right)$$

$$\Delta\tau_A = \int dt \left(1 - \frac{GM}{R} - \frac{1}{2}\bar{v}^2 \right);$$

TO GET THE ORBITAL SPEED OF ALICE, RECALL THAT THE VARIATION OF E ALONG AN ORBIT IS 0 SO:

$$0 = E_p + E_c \Rightarrow -E_p = E_c \Rightarrow v^2 = \frac{GM}{R}$$

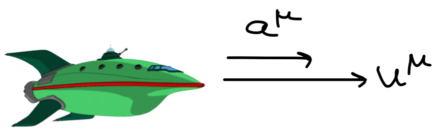
SO $$\Delta\tau_B = \Delta t \left(1 - \frac{GM}{R_E}\right)$$

$$\Delta\tau_A = \Delta t \left(1 - \frac{3}{2}\frac{GM}{R}\right)$$

$$\Delta\tau_B = \Delta\tau_A \Rightarrow \boxed{R = \frac{3}{2} R_E}$$ QED.

032 ROCKET PASSANGER

LET'S ASSUME THAT FRY, LEELA AND BENDER ARE TRAVELING ACROSS THE GALAXY:



① RECALL THAT $ds^2 = -d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu$;

$$-1 = \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau} ;$$

FOUR-VELOCITY FOR TIME-LIKE OBJECTS $\Rightarrow -1 = u_\mu u^\mu$

FOR THE ACCELERATION: $a^\mu = \dfrac{du^\mu}{d\tau}$ AND $a^\mu a_\mu = g^2$

ASSUMING THAT THE STATEMENT REFERS TO GRAVITY AS $g = 9.8 \frac{m}{s^2}$.

OBSERVE THAT THE OBJECT $a^\mu u_\mu = \eta_{\mu\nu} a^\mu u^\nu \Rightarrow$

$* \; d_\tau (\dot{x}^\mu \dot{x}^\nu) = (d_\tau \dot{x}^\mu) \dot{x}^\nu + d_\tau \dot{x}^\nu \dot{x}^\mu$

$$\Rightarrow \eta_{\mu\nu} \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) \left(\frac{dx^\nu}{d\tau}\right) = \frac{1}{2} \eta_{\mu\nu} \frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau} \cdot \frac{dx^\nu}{d\tau}\right) =$$

$$= \frac{1}{2}\eta_{\mu\nu}\frac{d}{d\tau}(u^\mu u^\nu) \underset{\uparrow}{=} \frac{1}{2}\, d/d\tau\,(\underbrace{u_\mu u^\mu}_{=-1}) = 0$$

BECAUSE η = FIXED

SO WE HAVE THAT :

$$\begin{cases} u^\mu u_\mu = -1 \\ a^\mu u_\mu = 0 \\ a^\mu a_\mu = g^2 \end{cases}$$

THIS SYSTEM WILL BE USEFUL FOR THE NEXT SECTION.

(2) TAKING THE ADVISE THAT THEY LIVE IN 2D, LET'S SOLVE THE PREVIOUS SYSTEM:

$$-u^{0\,2} + u^{1\,2} = -1 \implies u^{1\,2} = u^{0\,2} - 1 \; *$$

$$-a^0 u^0 + a^1 u^1 = 0 \rightarrow a^0 = \frac{u^1}{u_0} a^1 \; **$$

$$-a^{0\,2} + a^{1\,2} = g^2 \rightarrow \left(1 - \frac{u^{1\,2}}{u^{0\,2}}\right)^{**} a^{1\,2} = g^2$$

$$\implies \frac{u^{0\,2} - u^{0\,2} + 1}{u^{0\,2}}\, a^{1\,2} \, {}^{*} = g^2$$

$$\implies \boxed{a^1 = g\, u_0}$$

$a^0 = g u^1$

SO WE HAVE:

$$\begin{aligned} a^1 &= g u^0 \\ a^0 &= g u^1 \end{aligned} \Rightarrow \begin{aligned} \frac{du^1}{d\tau} &= g u^0 \xrightarrow{\text{DERIVE ONCE}} \frac{d^2 u^1}{d\tau^2} = g^2 u^1 \\ \frac{du^0}{d\tau} &= g u^1 \longrightarrow \frac{d^2 u^0}{d\tau^2} = g^2 u^0 \end{aligned} \quad \text{ODE}$$

$\Rightarrow u^i = A_i e^{g\tau} + B_i e^{-g\tau} \Rightarrow$ INTRODUCE IN $\dot{u}^1 = g u^0$ TO FIND $\{A_i, B_i\} \Rightarrow$

$$g A_1 e^{g\tau} - g B_1 e^{-g\tau} = g\left(A_0 e^{g\tau} + B_0 e^{-g\tau}\right)$$

$$\boxed{A_1 = A_0, \quad B_1 = -B_0}$$

TIME TO IMPOSE BOUNDARY CONDITIONS AND FIX $\{A_i, B_i\}$.

$$u^1\Big|_{\tau=0} = 1 \quad \rightsquigarrow \quad u^\mu = (-\tau, \vec{v}/c)$$

$$u^0\Big|_{\tau=0} = 0$$

SO $u^1 = 1$, $u^0 = 0 \Rightarrow$

$u^1 = 1 = A_1 + B_1 \Rightarrow A_1 = B_1 = 1/2$

$u^0 = 0 = A_0 + B_0 \Rightarrow A_0 = -B_0$

$$\boxed{\begin{aligned} &A_1 = A_0 = 1/2 \\ &B_1 = -1/2, \quad B_0 = 1/2 \end{aligned}}$$

SO : $u^1 = \frac{1}{2}\left(e^{-g\tau} + e^{-g\tau}\right) = \cosh(g\tau)$

$u^0 = \frac{1}{2}\left(e^{-g\tau} - e^{-g\tau}\right) = \sinh(g\tau)$

INTEGRATE :

$$\boxed{\begin{aligned} x^0(\tau) &= \frac{1}{g}\cosh(g\tau) + C \\ x^1(\tau) &= \frac{1}{g}\sinh(g\tau) + C \end{aligned}}$$

SO THIS MEANS THAT RELATIVISTIC OBJECTS FOLLOW HYPERBOLAS IN LORENTZIAN SPACE-TIMES.

OBSERVE : $-x_0^2 + x_1^2 = -\frac{1}{g^2}$; HYPERBOLOID EQ.

N33 SOME COORDINATE TRANSFORMATIONS

WE HAVE $\mathbb{R}^3$ (EUCLIDEAN; OJO, OJITO, OJAZO NOT LORENTIAN)

STARTING COORDINATES (x, y, z); MOVE TO CYLINDRIC (r, θ, z)

① THIS IS JUST THE JACOBIAN.

$$J = \begin{bmatrix} \frac{\partial x^0}{\partial y^0} & \frac{\partial x^1}{\partial y^0} & \cdots \\ \vdots & \frac{\partial x^1}{\partial y^1} & \vdots \\ \cdots & & \frac{\partial x^2}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & -r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det J = r^2\cos^2\theta + r^2\sin^2\theta = r^2.$$

② AN EUCLIDEAN ELEMENT IS THE WELL KNOW "DISTANCE".

$$g_{\mu\nu} = \mathbb{1}_{3\times 3} \quad \text{SO} \quad ds^2 = dx^2 + dy^2 + dz^2.$$

IN CYLINDRICAL $\quad ds^2 = [d(r\cos\theta)]^2 + [d(r\sin\theta)]^2 + dz^2 =$

$$ds^2 = (c\theta\, dr - r\sin\theta\, d\theta)^2 + (s\theta\, dr + r c\theta\, d\theta)^2 + dz^2 =$$

$$= dr^2 + r^2 d\theta^2 + dz^2.$$

③ LET'S ANALYSE THE SITUATION: THE HELIX GOES UP WITH CONSTANT RADIUS ?:

$$x(t) = \cos(t)$$
$$y(t) = \sin(t) \quad \Rightarrow \quad r^2 = x^2 + y^2 = 1$$

SO $r(t) = 1$. $z(t) = t$, SO $t \in \mathbb{R}$. FINALLY: θ CAN GO FROM $-\infty$ TO ∞, SO $\theta = t$.

④ WE KNOW THAT THE TANGENT VECTOR AT EACH POINT OF THE CURVE IS GIVEN BY:

$$\bar{t} = \frac{\bar{x}'}{|\bar{x}'|}$$

$\bar{x}'$ ← RESPECT TO t.

$|\bar{x}'|$ ← TO NORMALISE.

$\Rightarrow \bar{x}' = (-\sin t, \cos t, 1$

$|\bar{x}'| = \sqrt{2}$

MISSING Z...

$$\bar{t} = \left(-\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(-y/\sqrt{2}, x/\sqrt{2}, \frac{1}{\sqrt{2}}\right)$$

N51 TENSOR COORDINATE TRANSFORMATION

WE HAVE $S_{\mu\nu} = \begin{bmatrix} 1 & a \\ 0 & x^2 \end{bmatrix}$ IN 2D.

NEW COORD ARE $x' = \dfrac{2x}{y}$ AND $y' = y/2$;

WE JUST HAVE TO TRANSFORM THE TENSOR AS:

$$S_{\mu'\nu'} = \frac{\partial x^\alpha}{\partial x'^\mu} \cdot \frac{\partial x^\beta}{\partial x'^\nu} \cdot S_{\alpha\beta}$$

$$S_{00} = \frac{\partial x^\alpha}{\partial x'^0} \cdot \frac{\partial x^\beta}{\partial x^0} S_{\alpha\beta} = J S J^t \text{ with } J = \text{Jacobian.}$$

WRITE x AND y IN (x', y') DEPENCE AS....

$$y = 2y' \quad , \quad x = 2y'x'/2$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{bmatrix} = \begin{bmatrix} y' & x' \\ 0 & 2 \end{bmatrix} \Rightarrow$$

$$S_{\mu\nu}' = J^{T} S J = \begin{bmatrix} y' & x' \\ 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & a \\ 0 & \underbrace{x'^2 y'^2}_{x^2} \end{bmatrix} \begin{bmatrix} y' & 2 \\ x' & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} y'^2 & y'x' + 2ay' \\ x'y' & x'^2 + 2ax' + 4x'^2 y'^2 \end{bmatrix}$$

N23 FREE PARTICLE ACTION

WE ARE GIVEN THE FOLLOWING ACTION

$$S(\lambda) = -m \int d\lambda \sqrt{-g_{\mu\nu}(x)\,\dot{x}^{\mu}\dot{x}^{\nu}} \quad w/ \quad \dot{x}^{\mu} = \frac{dx^{\mu}}{d\lambda}$$

massive

① LET'S SAY THAT $\lambda \to s(\lambda)$ SO $d\lambda = \frac{d\lambda}{ds} \cdot ds$

FOR SURE THIS DOESNOT AFFECT, BECAUSE:

$$S = -m \int ds \, \frac{d\lambda}{ds} \sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{\frac{d\lambda}{ds}\cdot ds} \frac{dx^{\nu}}{\frac{d\lambda}{ds}\cdot ds}} =$$

$$S = -m \int ds \, \underbrace{\frac{\frac{d\lambda}{ds}}{\sqrt{\left(\frac{d\lambda}{ds}\right)^2}}}_{1} \sqrt{-g_{\mu\nu} \frac{dx^{\mu}}{ds} \cdot \frac{dx^{\nu}}{ds}} = \text{LIFE IS GOOD.}$$

② COMPUTE GEODESICS BY $\delta S = 0$

$$\delta S = -m \int d\lambda \; \delta\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} =$$

$$= -m \int d\lambda \; \frac{1}{2} \; \frac{+1}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} \cdot \delta\left(g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu\right) =$$

$$= -m \int d\lambda \; \frac{1}{2} \; \frac{1}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} \cdot 2 \cdot g_{\mu\nu} \dot{x}^\mu \delta \dot{x}^\nu \underset{IBP}{=}$$

$$= -m \int d\lambda \; \frac{d}{d\lambda}\left[\frac{1}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} g_{\mu\nu} \dot{x}^\mu\right] \delta x^\nu =$$

SO

$$\frac{\delta S}{\delta x^\nu} = \frac{d}{d\lambda}\left[\frac{1}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} g_{\mu\nu} \dot{x}^\mu\right] = 0$$

SO

$$-\frac{g_{\mu\nu}\ddot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} + g_{\mu\nu}\dot{x}^\mu \frac{-2\ddot{x}^\alpha \dot{x}_\alpha}{-2\sqrt{-\dot{x}^\mu \dot{x}_\mu}^{\,3}} = 0$$

OBSERVE THAT: IF $\lambda \equiv$ AFFIN PARAMETER $= \tau \Longrightarrow$

$\dot{x}^\mu = u^\mu \longrightarrow -\dot{x}^\mu \dot{x}_\mu = -u^\mu u_\mu = 1$ SO:

$$\boxed{\ddot{x}_\nu + \ddot{x}^\alpha \dot{x}_\alpha \dot{x}_\nu = 0}$$

N24 GEODESIC EQUATION

WE ARE GIVEN A LINE INVARIANT OF THE FORM:

$$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right)$$

THERE ARE TWO WAYS TO COMPUTE THIS:

- COMPUTE CHRISTOFFEL TO GET GEODESIC $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda = 0$
- USE EULER - LAGRANGE TO GET EOM $\equiv$ GEODESIC.

FOR SIMPLICITY, CHOOSE THE SECOND ONE.

OBSERVE THAT $L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \begin{cases} \bullet\ 0 & \text{IF } \omega = 0 \equiv \text{LIGHT-LIKE} \\ \bullet\ -1 & \text{IF } \omega \neq 0 \equiv \text{TIME-LIKE} \end{cases}$

WE HAVE TIME LIKE SO, $L = -1$;

$$L = \frac{1}{t^2}\left(-\dot{t}^2 + \dot{x}^2\right)$$

2 EOM (t, x): $\quad \dfrac{d}{dz}\dfrac{dL}{d\dot{x}^i} - \dfrac{dL}{dx^i} = 0$

$$\textcircled{t} \quad \frac{d}{d\tau}\left(\frac{-2\dot{t}}{t^2}\right) - \frac{2\dot{t}^2}{t^3} = 0$$

$$\frac{-2\ddot{t}\,t^2 + 4\dot{t}^2 t}{t^4} - \frac{2\dot{t}^2}{t^3} = -\ddot{t} + \frac{\dot{t}^2}{t} = 0 \,;$$

$$\textcircled{x} \quad \frac{d}{d\tau}\left(\frac{2\dot{x}}{t^2}\right) - 0 = 0$$

OJO. OJITO. OJAZO   CONSERVED QUANTITY $\frac{\dot{x}}{t^2} = p$

USE INSIDE LAGRANGIAN:

$$L = -1 = -\frac{\dot{t}^2}{t^2} + t^2 p^2 \Rightarrow -t^2 = -\dot{t}^2 + t^4 p^2 \Rightarrow$$

$$\dot{t} = t\sqrt{1 + t^2 p^2}$$

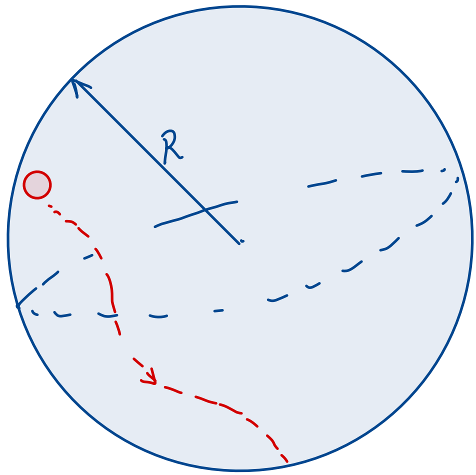
TO OBTAIN THE $x(t)$ GEODESIC, OBSERVE THAT:

$$\frac{dx}{dt} = \frac{dx}{d\tau} \cdot \frac{d\tau}{dt} = \frac{\dot{x}}{\dot{t}} = \frac{p t^2}{t\sqrt{\cdots}} =$$

$$\boxed{x(t) = \int \frac{dx}{dt} \cdot dt \Rightarrow \frac{1}{p}\sqrt{1 + (pt)^2} + k}$$

N31 FREE PARTICLE ON A SPHERE.

SO WE HAVE SOME SORT OF 2D SIMPLE PENDULUM PROBLEM. (WITH GEOMETRY)



AS WE ARE CONFINED TO A 2D SURFACE, SPHERICAL ONE, WE KNOW!

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) = d\Omega_2^2$$

$$\Rightarrow g_{\mu\nu} = \mathrm{Diag} \{ R^2 , R^2 \sin^2\theta \}.$$

THE LAGRANGIAN? $L = T - \cancel{V}$ NOT HERE

$$= \frac{1}{2} m \dot{\vec{x}}^2 = \frac{1}{2} m R^2 \left(\frac{d}{dt} \left(\cos\theta \sin\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\phi \, \hat{z} \right) \right)^2$$

$$= \cdots = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right)$$

RECALL THAT $L = \frac{1}{2} m \, g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}$

$$L = \frac{1}{2} m R^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \right) \Rightarrow \text{SAME THING AS BEFORE}$$

(b) FIND EOM AND Γ:

EOM $\Rightarrow$ 2 DOF (θ, ϕ)

(θ) $\partial_\tau \left(m R^2 \dot{\theta} \right) - \frac{m R^2}{2} 2 \sin\theta \cos\theta \, \dot{\phi}^2 = 0$

(φ) $\partial_\tau \left(m R^2 \sin^2\theta \, \dot{\phi} \right) - 0 = 0 \Rightarrow$ OJO, OJITO, OJAZO CONSERVED QUANTITY.

$$\Rightarrow \ddot{\theta} - \sin\theta \cos\theta \, \dot{\phi}^2 = 0$$

$$\ddot{\phi} + 2 \operatorname{ctg}\theta \; \dot{\theta} \dot{\phi} = 0$$

THE CHRISTOFFEL: $\Rightarrow g^{\mu\nu} \Rightarrow (g_{\mu\nu})^{-1}$ OJO. B.C IT IS DIAGONAL!

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \partial_\theta g_{\theta\theta} = 0$$

$$\Gamma^{\theta}_{\phi\theta} = \Gamma^{\theta}_{\theta\phi} = \frac{1}{2} g^{\theta\alpha} \left(\partial_\alpha g_{\theta\phi} + \partial_\theta g_{\phi\alpha} - \partial_\phi g_{\alpha\theta} \right) = 0$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} g^{\theta\alpha} \left(\partial_\phi g_{\phi\alpha} + \partial_\phi g_{\alpha\phi} - \partial_\alpha g_{\phi\phi} \right) =$$

$$= -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\phi\phi} = -\sin\theta \cos\theta$$

$$\Gamma^{\phi}_{\phi\phi} = 0$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\alpha} \left(\partial_\theta g_{\alpha\phi} + \dots \right) =$$

$$= +\frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} = +\mathrm{ctg}\,\theta$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\alpha} \left(\partial_\alpha g_{\theta\theta} + \partial_\theta g_{\alpha\theta} - \partial_\theta g_{\theta\alpha} \right) = 0$$

CHECK GEODESIC:

θ) $$\ddot{x}^{\theta} + \Gamma^{\theta}_{\nu\lambda} \dot{x}^{\nu} \dot{x}^{\lambda} = 0$$

$$\ddot{\theta} - \sin\theta \cos\theta\, \dot{\phi}^2 = 0$$

ϕ) $$\ddot{x}^{\phi} + \Gamma^{\phi}_{\nu\lambda} \dot{x}^{\nu} \dot{x}^{\lambda} = 0$$

$$\ddot{\phi} + \underbrace{\Gamma^{\phi}_{\theta\phi} \dot{\theta}\dot{\phi} + \Gamma^{\phi}_{\phi\theta} \dot{\phi}\dot{\theta}} = \ddot{\phi} + \mathrm{ctg}\,\theta\, \dot{\phi}\dot{\theta} = 0$$

OJO; WHEN $\Gamma^{\alpha}_{\beta\alpha}$ ADD A 2 DUE TO SYM.

OBSERVE THAT GEODESIC ARE THE EOM $\Leftarrow$ IMPORTANT.

CHALLENGE $\rightarrow$ DO IT FOR D DIM; $ds^2 = R^2 \left(d\theta_1^2 + \sum_j^D \left(\prod_i^j \sin^2\theta_i \right) d\theta_i^2 \right)$

SPOILER $\Rightarrow$ SYMMETRIES.

N43 Geodesic on a disk.

We are given the following metric:

$$ds^2 = dr^2 + \frac{r^2}{1-r^2\omega^2}\, d\theta^2$$

We refer to the spatial components and we want to find the shortest paths on it. Assume an affine parameter.

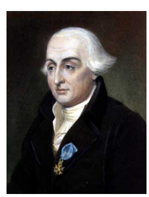
(1) EOM ≡ Geodesics

$$L = \dot{r}^2 + \frac{r^2}{1-r^2\omega^2}\,\dot{\theta}^2$$

$$r)\quad 2\ddot{r} + \frac{2r(1-r^2w^2) - r^2(-w^2 2r)}{(1-r^2w^2)^2}\,\dot{\theta}^2 =$$

$$\ddot{r} + \frac{r}{(1-r^2w^2)^2}\,\dot{\theta}^2$$

$$\theta)\quad \frac{d}{d\tau}\Bigg(\underbrace{\frac{2r^2}{(1-r^2w^2)}\,\dot{\theta}}_{\text{CONSERVED QUANTITY}}\Bigg) = 0$$

OMG  !

$$\alpha = \frac{r^2}{1-r^2\omega^2}\,\dot{\theta}$$, WHICH MEANS... $\Longrightarrow$

(2) $$\boxed{\frac{d\theta}{ds} = \frac{1-r^2\omega^2}{r^2}\,\alpha}$$ ✓

TAKE THIS AND PUT INSIDE $L = 1$ (MASSIVE) TO SEE:

$$L = 1 = \dot{r}^2 + \frac{r^2}{1-r^2\omega^2}\,\dot{\theta}^2 \Rightarrow$$

$$\dot{r}^2 = 1 - \frac{r^2}{1-r^2\omega^2}\,\frac{(1-r^2\omega^2)^2}{r^4}\,\alpha^2 \Rightarrow$$

$$\dot{r} = \pm\sqrt{1 - \frac{1}{r^2}\alpha^2 + \omega^2\alpha^2} = \boxed{\pm\sqrt{\beta - \frac{\alpha^2}{r^2}} = \frac{dr}{d\tau}}$$

FINALLY :

$$\frac{dr}{d\theta} = \frac{dr}{d\tau}\cdot\frac{d\tau}{d\theta} = \boxed{\frac{\dot{r}}{\dot{\theta}} = \pm\frac{r^2\sqrt{\beta - \alpha^2/r^2}}{(1-r^2\omega^2)\,\alpha}}$$

(3) ASSUME $\left.\dot{r}\right|_{r=r_0} = 0 \Rightarrow 1 + \alpha^2\left(\omega^2 - \frac{1}{r_0^2}\right) = 0$

$$\Rightarrow \alpha = \pm\sqrt{\frac{1}{\frac{1}{r_0^2} - \omega^2}}$$

(4) AS w^2 IS A FIXED QUANTITY, IF $\alpha \to 0$ MEANS $r_0 \to 0$, WITH $r_0 = 0$ THE CENTER OF THE ROTATING DISK. ALSO $\alpha \to 0 \Rightarrow \dot{\theta} \simeq 0$ AND $\dot{r} \simeq \sqrt{\beta}$

SO $r(t) = \underbrace{\sqrt{\beta}}_{=1}\, t + r_0$, WHICH IS A LINE.

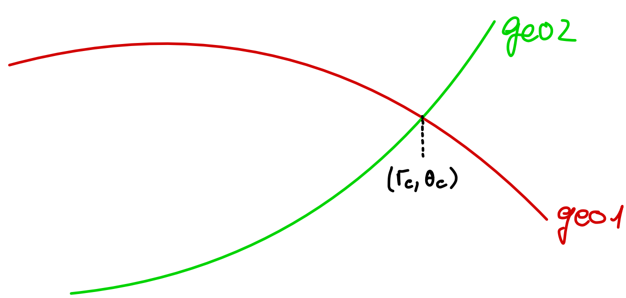
(5) SHOW THAT THE GEODESIC CROSS $r_* = \frac{1}{w}$ AT $90°$.

$A = (r_*, 0)$. THIS MEANS THAT $(\dot{r}, \dot{\theta})$ SHOULD BE $\mathring{A} = (1, 0) \Rightarrow$ PLUG r_* INSIDE $\dot{r}, \dot{\theta}$ THEN:

$$\dot{r}\Big|_{r_*} = \pm\sqrt{1 - w^2\alpha^2 + \alpha^2 w^2} = 1 \checkmark$$

$$\dot{\theta}\Big|_{r_*} = \frac{1 - \frac{1}{w^2} w^2}{1/w^2}\,\alpha = 0 \checkmark$$

(6) LET'S ASSUME THE FOLLOWING:



AT THE POINT (r_c, θ_c) WE HAVE TWO GEODESICS AND THEIR RESPECTIVE TANGENT VECTORS $T_i = (\dot{r}_i, \dot{\theta}_i)$. WE ALSO KNOW THAT THE ANGLE BTW TWO VECTORS IS:

$$\cos\phi = \frac{\langle T_i, T_j \rangle}{|\langle T_i, T_i \rangle| \, |\langle T_j, T_j \rangle|}$$

WHERE $\langle \circ, \circ \rangle$ IS THE SCALAR PRODUCT GIVEN BY THE METRIC OF THE SPACE. OBSERVE $\langle T_i, T_i \rangle = g_{\mu\nu} \dot{T}_i^{\mu} \dot{T}_i^{\nu} = L = 1$!

AND $\langle T_i, T_j \rangle =$

$$= 1 \cdot \dot{r}_1 \dot{r}_2 + \frac{r^2}{1 - r^2\omega^2} \dot{\theta}_1 \dot{\theta}_2 = \cos\phi .$$

BUT WE HAVE THE VALUES OF GENERIC $(\dot{r}, \dot{\theta}) \Rightarrow$

$$= \sqrt{\beta - \frac{\alpha_1^2}{r^2}} \sqrt{\beta - \frac{\alpha_2^2}{r^2}} + \frac{r^2}{1 - r^2\omega^2} \cdot \frac{(1 - r^2\omega^2)^2}{r^4} \alpha_1 \alpha_2$$

OJO. OBSERVE THAT EACH GEODESIC WILL HAVE A DIFFERENT ASSOCIATED α_i.

052 EXPANDING UNIVERSE

WE WANT TO STUDY THE GEODESICS OF AN FLRW UNIVERSE, DESCRIBED BY:

$$ds^2 = -dt^2 + \underset{}{\overset{a^2}{\overset{\parallel}{f(t)}}}\, \eta_{ij}\, dx^i dx^j .$$

THIS METRIC IS GOING TO BE CRUCIAL WHEN WE EXPLORE COSMOLOGY...

AS ALWAYS!

$$g_{\mu\nu} = \{-1, f, f, f\}$$

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\lambda g_{\sigma\nu} + \partial_\nu g_{\lambda\sigma} - \partial_\sigma g_{\nu\lambda} \right)$$

BECAUSE DIAGONAL METRIC.

$$\Gamma^{t}_{tt} = \frac{1}{2} g^{tt} \partial_t g_{tt} = 0$$

$$\Gamma^{t}_{ti} = \frac{1}{2} g^{tt} \left(\partial_t g_{ti} + \partial_i g_{tt} - \partial_t g_{it} \right) = 0$$

$$\Gamma^{t}_{ij} = \frac{1}{2} g^{tt} \left(\partial_j g_{ti} + \partial_i g_{jt} - \partial_t g_{ij} \right) = -\frac{1}{2} g^{tt} \partial_t g_{ij}$$

$$\Gamma^i_{tt} = \frac{1}{2} g^{ii} \left(\partial_t g_{it} + \partial_t g_{ti} - \partial_i g_{tt} \right) = 0$$

$$\Gamma^i_{tj} = \frac{1}{2} g^{ii} \left(\partial_t g_{ij} + \partial_j g_{ti} - \partial_i g_{tj} \right)$$

$$= \frac{1}{2} g^{ii} \partial_t g_{ii}$$ BECAUSE DIAGONAL

$$\Gamma^i_{jk} = \frac{1}{2} g^{ii} \left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk} \right) = 0$$

SO:

$$\Gamma^t_{ii} = -\frac{1}{2} g^{tt} \partial_t g_{ii}$$

$$\Gamma^i_{ti} = \frac{1}{2} g^{ii} \partial_t g_{ii}$$

THE GEODESICS ARE: $a = \{t, x, y, z\}$

$$\ddot{a}^\mu + \Gamma^\mu_{\nu\lambda} \dot{a}^\nu \dot{a}^\lambda = 0 :$$

- $$\ddot{t} + \Gamma^t_{ij} \dot{a}^i \dot{a}^j = \ddot{t} + \Gamma^t_{xx} \dot{x}^2 + \Gamma^t_{yy} \dot{y}^2 + \Gamma^t_{zz} \dot{z}^2 =$$

$$= \ddot{t} - \frac{1}{2} g^{tt} \partial_t f \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) =$$

$$= \ddot{t} + \frac{1}{2} (1) \dot{f} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

- $\ddot{a}^x + \Gamma^x_{\nu\lambda}\, \dot{a}^\nu \dot{a}^\lambda = \ddot{x} + 2\Gamma^x_{tx}\, \dot{t}\dot{x} =$

$$= \ddot{x} + \dot{f}/f\; \dot{t}\dot{x} = 0$$

SAME FOR $y, z \ldots$

SO WE HAVE :

- $\ddot{t} + \dot{a}a\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0$
- $\ddot{i} + \dfrac{2\dot{a}}{a}\,\dot{t}\dot{i} = 0$ WITH $i = \{x, y, z\}$.

② FOR A NULL GEODESIC, WE CAN USE THE LAGRANGIAN AS!

$$L = -\dot{t}^2 + a^2(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 0 \text{ FOR PHOTONS.}$$

AS $y, z = cte$.

$$0 = -\dot{t}^2 + a^2\dot{x}^2 \implies \boxed{\pm \int \frac{dt}{a(t)} = x(t) + c}$$

053 MORE GEODESICS

GIVEN A NON-DIAGONAL METRIC AS:

$$ds^2 = \frac{z^2}{R^2} dx^2 + dz^2 - 2dxdy = g_{\mu\nu} dx^\mu dx^\nu$$

OJO. OJITO. OJAZO OBSERVE IT IS NON DIAGONAL; THIS SHOULD MAKE US RAISE AN EYEBROW... WE WILL FACE PROBLEMS. BE AWARE!

① METRIC:

$$g_{\mu\nu} = \begin{array}{cc} \begin{array}{ccc} x & y & z \end{array} & \\ \begin{bmatrix} z^2/R^2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} x \\ y \\ z \end{array} \end{array}$$

OBSERVE THAT 2 ACCOUNTS FOR SYM IN $xy = yx$. (TORSION FREE)

THE INVERSE $\Rightarrow$

$$g^{\mu\nu} = \frac{|\quad|^t}{\mathrm{Det}[g]} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -z^2/R^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

② LET'S FIND THE GEODESICS AND CHRISTOFFELS IN THE SAME RUN. RECALL THAT THE GEODESICS ARE THE EOM; THESE EOM POP UP FROM THE VARIATION OF THE ACTION, RESPECT TO EACH CANONICAL COORDINATE.

$$S=\int L\,d\tau = \int g_{\mu\nu}\,\dot{x}^\nu \dot{x}^\mu\, d\tau \;;\quad \text{AND}\quad \frac{\delta S}{\delta x^\mu}=0.$$

$$S+\delta_x S = \int -2\frac{d}{d\tau}(x+\delta x)\frac{d}{d\tau}y - \frac{1}{R^2}z^2\left(\frac{d}{d\tau}(x+\delta x)\right)^2 + \left(\frac{dz}{d\tau}\right)^2 d\tau =$$

$$= \int -2\dot x\dot y - 2\, d_\tau\delta x\, d_\tau y - \frac{1}{R^2}z^2\left[\left(\left(\frac{dx}{d\tau}\right)^2 + \cancel{\delta^2\left(\frac{dx}{d\tau}\right)^2} + 2\frac{dx}{d\tau}\left[\delta\frac{dx}{d\tau}\right]\right)\right]$$

SECOND ORDER

$$+\left(\frac{dz}{d\tau}\right)^2 d\tau \;;$$

$$= S + \int -2\, d_\tau\delta x\, d_\tau y - \frac{1}{R^2}z^2\; 2\left(\frac{dx}{d\tau}\right)\frac{d\delta x}{d\tau} =$$

$$= S + \int -2\left[\cancel{\frac{d}{d\tau}(\delta x\, d_\tau y)} - \frac{d^2}{d\tau^2}y\,\delta x\right] -$$

Boundary

$$-\frac{2}{R^2}\left[\cancel{\frac{d}{d\tau}\left(z^2\frac{dx}{d\tau}\delta x\right)} - \left(\frac{d}{d\tau}z^2\right)\frac{dx}{d\tau}\cdot\delta x - z^2\frac{d^2x}{d\tau^2}\delta x\right]$$

BOUNDARY

$$\delta_x S = 0 = \ddot{y} + \frac{2z}{R^2}\dot{z}\dot{x} + 2z^2\ddot{x} = 0$$

$\Gamma^y_{zx} = \Gamma^y_{xz}$

Similarly:

$$\delta_y S = \ddot{x} = 0$$

$$\delta_z S = 0 = \ddot{z} + \frac{z}{R^2}\dot{x}^2$$

Γ^z_{xx}

③ TO FIND CONSERVED QUANTITIES IS JUST AS "EASY" AS TO LOOK AT THE METRIC ENTRIES AND IDENTIFY WHICH DIMENSION VARIABLES ARE NOT PRESENT (aka KILLING VECTORS)

IN OUR CASE $\{x, y\}$ THERE SHOULD BE SOMETHING THERE.

OBSERVE THAT $\ddot{x} = 0 \Rightarrow \dot{x} = c$

ALSO: $\ddot{y} + \frac{2z}{R^2}\dot{z}c \Rightarrow \frac{d}{d\tau}\left(\dot{y} + \frac{z^2}{R^2}c\right) = 0$

SO $\dot{y} + \frac{z^2}{R^2}c = k \Rightarrow \boxed{\dot{y} + \frac{z^2}{R^2}\dot{x} = k}$

(4) FIND THE CURVES $z(x)$ AND $y(x)$. OBSERVE Z-GEO.

$$\ddot{z} + \frac{z}{R^2}\dot{x}^2 = 0 \quad \Longrightarrow \quad \ddot{z} + \frac{c}{R^2} z = 0 \; *$$

$$\Rightarrow \frac{dz}{d\tau} = \frac{dz}{dx}\cdot\frac{dx}{d\tau} = \frac{dz}{dx}\cdot c$$

$$\frac{d}{d\tau}\left(\frac{dz}{dx}\cdot c\right) = c\cdot\frac{d}{d\tau}\left(\frac{dz}{dx}\right) = \frac{dx}{d\tau}\cdot\frac{d^2z}{dx^2}\cdot c =$$

$$= c^2 z''$$

SO $*$ $= c^2 z'' + \frac{c}{R^2} z = 0 \Rightarrow$ $\boxed{z(x) = A\cos\left(\frac{x}{\sqrt{c}R} + \phi_0\right)}$

SAME TRICK FOR $y(x)$; RECALL THAT:

$$\dot{y} + \frac{z^2}{R^2}\dot{x} = k \; ; \qquad \frac{dy}{d\tau} = \frac{dy}{dx}\cdot\frac{dx}{d\tau}$$

$$cy' + z^2/R^2\, c = k :$$

$$y' = \frac{k}{c} - \frac{1}{R^2}A^2\cos^2\left(\frac{x}{\sqrt{c}R} + \phi_0\right) =$$

$1/2 + \frac{1}{2}\cos 2m$

$$y = \int dx = \frac{k}{c}x - \frac{A^2}{R^2}\int dx\, \cos^2(\;) =$$

$$= \boxed{\left(\frac{k}{c} - \frac{A^2}{2R^2}\right)x - \frac{A}{4R\sqrt{c}}\sin\left(\frac{2x}{\sqrt{c}R} + \phi_0\right)}$$

Riemman Curvature and Einstein Equations

Riemann Tensor Identities

1. Prove the following identities for the Riemann Tensor associated to the Levi Civita connection (Hint: you should use normal coordinates):
   (a) $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$,
   (b) $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$,
   (c) $R_{\mu[\nu\rho\sigma]} = 0$,
   (d) $R_{\mu\nu[\rho\sigma;\tau]} = 0$.
2. Use (d) to prove the contracted Bianchi identity: $\nabla^a G_{ab} = 0$.

Riemman Tensor Properties and Symmetries

Using the symmetries of the Riemann Tensor $R_{\mu\nu\rho\sigma}$ ((a)-(c) of previous exercise) compute the number of independent components in 2,3 and 4 dimensions. Try to extend this to general dimensions D.

Some Bianchi identities

Show that for a given antisymmetric tensor $A_{\mu\nu}$ the following relation holds

$$\nabla_{[\mu}A_{\nu\lambda]} = \partial_{[\mu}A_{\nu\lambda]}. \tag{1}$$

Symmetries of Riemann tensor

Consider a Riemann tensor of the form

$$R_{\mu\nu\sigma\kappa} = \lambda\left(g_{\mu\sigma}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\sigma}\right). \tag{2}$$

1. Show that this expression have the proper symmetries.
2. Find the Ricci tensor, Ricci scalar and Einstein tensor.
3. Show that the Einstein tensor satisfies the Bianchi identity.

Curvature in 2 dimensions

In 2 dimensions, there is only one independent component of the curvature tensor, say R_{1212} (this one component is equivalent to the Gauss curvature of a 2 -surface). As a consequence, there must be a simple relation between R_{1212} and the scalar curvature.

1. Using the definition for Ricci scalar, the antisymmetry of the Riemann tensor in its first and second pair of indecies, and the fact that in two dimension the inverse metric is explicitly given by

   $$g^{\alpha\beta} = \frac{1}{g_{11}g_{22} - g_{12}g_{21}}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{pmatrix}. \tag{3}$$

   Show that the Riemann tensor and the Ricci scalar are related by

   $$R = \frac{2}{g_{11}g_{22} - g_{12}g_{21}}R_{1212} \tag{4}$$

2. Calculate the scalar curvature of the metric $ds^2 = dx^2 + e^{2x}dy^2$.

Curvature in $2+1$ dimensions

By counting the number of independent components of the Riemann tensor, show that the Einstein equations in the empty space imply $R_{\mu\nu\rho\lambda} = 0$ in $2+1$ dimensions.

Curvature in $3+1$ dimensions

Consider the space-time metric

$$ds^2 = -dt^2 + dz^2 + f^2(z)\left(dr^2 + r^2 d\theta^2\right). \tag{5}$$

where β is a constant.

1. Find the nonzero Christoffel symbols for this metric.
2. Find the nonzero components of the Riemann tensor.
3. Find the Ricci tensor, Ricci scalar.

More Riemmanian Computations

Consider the metric

$$ds^2 = \left(-dt^2 + dx^2 + dy^2\right)e^{2\beta z} + dz^2. \tag{6}$$

1. Find the Christoffel symbols.
2. Find the Riemann tensor.
3. Find the Ricci tensor and scalar.
4. Find the Einstein tensor.

Curvature inside a Compact Space

Conifolds are 6D important manifolds in several research lines within theoretical physics. Their base is called $T_{1,1}$ and it is a 5D compact space (five angles that start at $\rho = 0$ (the tip of the conifold) and grows up in volume towards $\rho \to \infty$). In order to practice how to compute Riemanns and Riccis, we do not need to know about its features, but just to look at its metric, that looks like:

$$ds_6^2 = \kappa^{-1}(\rho)d\rho^2 + \frac{1}{9}\kappa(\rho)\rho^2 e_\psi^2 + \frac{1}{6}\rho^2\left(e_{\theta_1}^2 + e_{\phi_1}^2\right) + \frac{1}{6}\left(\rho^2 + 6a^2\right)\left(e_{\theta_2}^2 + e_{\phi_2}^2\right), \tag{7}$$

Where:

$$e_\psi = d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i, \quad e_{\theta_i} = d\theta_i, \quad e_{\phi_i} = \sin\theta_i d\phi_i, \quad i = 1,2, \tag{8}$$

and

$$\kappa(\rho) \equiv \frac{\rho^2 + 9a^2}{\rho^2 + 6a^2} \qquad a \epsilon \mathbb{R}. \tag{9}$$

With plenty of patience, time and a good coffee (and/or beer) by your side, do:

1. Obtain $g_{\mu\nu}$ when $\rho \to 0$ and $\rho \to \infty$.
2. Compute its Christoffel symbols.
3. Calculate non-zero Riemann tensors, Ricci and Ricci scalar.

Tip: This could be a nice chance for you to use your coding skills and code some lines to compute this for you.

Normal Vectors in Minkowski Spacetime

In this exercise we will study more in detail normal vectors to hypersurfaces in flat spacetime. In particular, we consider the Minkowski metric in (t, r, θ, ϕ) coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{10}$$

In the rest of the exercise, we will study the hypersurface $r = \text{const}$.

1. Find the normal vector to the hypersurface in (t, r, θ, ϕ) coordinates, and give a basis for the tangent vectors.
2. Define new coordinates $\hat{v} = t + r, \hat{r} = r$ and repeat the analysis. Is the radial coordinate still the same?

Killing Vectors

Find a complete set of Killing vector fields for the following spaces:

1. Minkowski space with metric

   $$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{11}$$

   How many Killing vectors are there? Provide their physical interpretation.
2. Rindler space with metric

   $$ds^2 = -r^2 dt^2 + dr^2 \tag{12}$$

Parallel Transport

Consider the 2 -sphere with the usual round metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Consider a point p on the equator and a vector X_p pointing in the θ direction. Parallel transport the vector an angle ϕ_0 in the ϕ direction, then up to the North Pole, then back to p. Denote the new vector at p as X'_p. What is the angle between X_p and X'_p ?

Conformal transformations

A conformal transformation of a space-time is one where the metric $g_{\mu\nu}$ of an original space-time is transformed into the metric $\tilde{g}_{\mu\nu}$ of a new space-time such that the two metrics are related as follows

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \tag{13}$$

where Ω is a function of the space-time coordinates x^μ.

1. Suppose in the old space-time one has a solution to the source-free Maxwell's equations

   $$\nabla_\mu F^{\mu\nu} = 0 \quad \text{and} \quad \nabla_{[\mu} F_{\nu\lambda]} = 0, \tag{14}$$

   with F being the antisymmetric field strength tensor. Show that $F_{\mu\nu}$ is also a solution to these equations in the new space-time with metric $\tilde{g}_{\mu\nu}$.
2. The metric of a $\kappa = 0$ Robertson-Walker space-time is sometimes written as

   $$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right). \tag{15}$$

   Show that this space-time is conformal to Minkowski space-time.

Covariant derivative for Physicists

1. Consider a general co-vector ω_ν. Does $\partial_\mu \omega_\nu$ transform like a $(0,2)$ tensor? We will try to fix this problem by introducing a new "derivative" which transforms like a tensor. Consider the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right). \tag{16}$$

2. Use the tensorial behavior of the metric under a coordinate transformation to show that the Christoffel symbols transform as

$$\Gamma^{\mu'}_{\nu'\lambda'} = \Gamma^\mu_{\nu\lambda} \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} + \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial x^{\nu'} \partial x^{\lambda'}}. \tag{17}$$

3. Consider now a "derivative" of the form

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu} \omega_\rho. \tag{18}$$

   Does it transform like a tensor?

Covariant derivative for Mathematicians

A covariant derivative ∇ is a map sending a vector field X and a (p,q) tensor T to another (p,q) tensor $\nabla_X T$, which has the interpretation of "derivative" of T along the curve defined by X. In other words, ∇T is a $(p,q+1)$ tensor such that $\nabla T(X,\ldots) = \nabla_X T(\ldots)$. Suppose we have a coordinate basis $\{e_\mu\}$ for vectors and $\{\theta^\nu\}$ for 1-forms. Using the properties of ∇ prove that:

1. $\nabla_\mu X^\nu = \partial_\mu X^\nu + \Gamma^\nu_{\rho\mu} X^\rho$,
2. $\nabla_\mu \eta_\nu = \partial_\mu \eta_\nu - \Gamma^\rho_{\nu\mu} \eta_\rho$. Note that by definition $\nabla_{e_\mu} e_\nu = \Gamma^\rho_{\nu\mu} e_\rho, \nabla_\mu X^\nu = \nabla X\left(e_\mu, \theta^\nu\right)$ and $\nabla_\mu \eta_\nu = \nabla \eta\left(e_\mu, e_\nu\right)$. Use the results (1) and (2) to find $\nabla_\mu T^{\alpha_1 \ldots \alpha_p}\beta_1 \ldots \beta_q$ for a general (p,q) tensor T.

Geodesic Deviation

Consider a family of geodesics forming a two-dimensional surface in spacetime. We can assign coordinates (t,s) such that $T = \partial_t$ is geodesic, and $S = \partial_s$. Hence we have a geodesic for each value of s. Note that $[T,S] = 0$. We are interested in the behaviour of neighbouring geodesics, e.g. whether they will move away or towards each other. At some value of t we have $x^\mu(t, s+\delta s) = x^\mu(t,s) + \delta s S^\mu$ hence $\delta s S^\mu$ is the relative position vector between two neighbouring geodesics. We can consider $\nabla_T \nabla_T S$, a kind of relative acceleration. Show that for a torsion-free connection:

$$T^a \nabla_a \left(T^b \nabla_b S_c \right) = R_{cdab} T^a T^d S^b. \tag{19}$$

This is known as the geodesic deviation equation. (Recall that the torsion tensor is defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and the Riemann tensor by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$)

Challenge Problem

Let ∇ be the covariant derivative associated with a connection that is not torsion free. Let $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ where X and Y are vector fields. Show that this defines a $(1,2)$ tensor field T. This is called the torsion tensor. Find its components $T_{\mu\nu}$ in a coordinate basis. Show that $2\nabla_{[a}\nabla_{b]} f = -T^c_{ab} \nabla_c f$, where f is any function.

Challenge Problem (Bis)

In a spacetime of n dimensions define a tensor

$$C_{abcd} = R_{abcd} + \alpha\left(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}\right) + \beta R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right), \tag{20}$$

where α and β are constants. Show that C_{abcd} has the same symmetries as R_{abcd}. How do the coefficients α and β have to be chosen to set $C^{a}_{bad} = 0$? With this extra condition, C_{abcd} is called the Weyl tensor. Show that it vanishes if n = 2, 3. Setting n = 4, how many independent components do R_{ab} and C_{abcd} have? What does the Weyl curvature represent physically? Show that in vacuum

$$\nabla^{a} C_{abcd} = 0. \tag{21}$$

N62 RIEMMAN TENSOR IDENTITIES

WHEN WE SEE A RIEMMAN TENSOR FULLY COVARIANT (4 INDEX $\downarrow$) IT IS BETTER TO EXPRES IT AS:

$$R_{\alpha\mu\nu\kappa} = \frac{1}{2}\left[\partial_{\kappa\mu} g_{\alpha\nu} - \partial_{\kappa\alpha} g_{\mu\nu} - \partial_{\nu\mu} g_{\alpha\kappa} + \partial_{\nu\alpha} g_{\mu\kappa}\right]$$
$$+ g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\alpha}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\alpha}\Gamma^{\sigma}_{\mu\nu}\right]$$

(1)

a) SO INDEXES 1,2 BECOME 3,4 AND VICEVERSA. TAKE RHS OF PREVIOUS EXPRESION AND SUBSTITUTE:

$$\Rightarrow \frac{1}{2}\left(\partial_{\mu\kappa}\partial_{\nu\alpha} - \partial_{\mu\nu} g_{\kappa\alpha} - \partial_{\kappa\alpha} g_{\nu\mu} + \partial_{\alpha\nu} g_{\kappa\mu}\right)$$
$$+ g_{\eta\sigma}\left[\Gamma^{\eta}_{\alpha\nu}\Gamma^{\sigma}_{\kappa\mu} - \Gamma^{\eta}_{\alpha\kappa}\Gamma^{\sigma}_{\nu\mu}\right] = R_{\nu\kappa\alpha\mu}. \checkmark$$

b) $\nu \leftrightarrow \kappa \Rightarrow \frac{1}{2}\left[\partial_{\nu\mu} g_{\alpha\kappa} - \partial_{\nu\alpha} g_{\mu\kappa} - \partial_{\kappa\mu} g_{\alpha\nu} + \partial_{\kappa\alpha} g_{\mu\nu}\right]$ $\checkmark$

$$+ g_{\eta\sigma}\left[\Gamma^{\eta}_{\kappa\alpha}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\eta}_{\nu\alpha}\Gamma^{\sigma}_{\mu\kappa}\right] = -R_{\alpha\mu\nu\kappa} = R_{\alpha\mu\kappa\nu}$$

(c) $R_{\lambda[\mu\nu\kappa]} = 0$:

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} + R_{\lambda\kappa\mu\nu} - R_{\lambda\mu\kappa\nu} - R_{\lambda\kappa\nu\mu} - R_{\lambda\mu\nu\kappa} = 0$$

BUT $R_{\lambda\mu\nu\kappa} = - R_{\lambda\mu\kappa\nu} \Longrightarrow$

$$2\left(R_{\lambda\mu\nu\kappa} + R_{\lambda\nu\kappa\mu} + R_{\lambda\kappa\mu\nu}\right) = 0$$

$$\begin{aligned}
R_{\lambda\mu\nu\kappa} = {} & \frac{1}{2}\left[\partial_{\kappa\mu} g_{\lambda\nu} - \partial_{\kappa\lambda} g_{\mu\nu} - \partial_{\nu\mu} g_{\lambda\kappa} + \partial_{\nu\lambda} g_{\mu\kappa}\right] \\
& + g_{\eta\sigma}\left[\Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\mu\nu}\right] + \\
& \frac{1}{2}\left[\partial_{\nu\mu} g_{\lambda\kappa} - \partial_{\mu\lambda} g_{\nu\kappa} - \partial_{\kappa\nu} g_{\lambda\mu} + \partial_{\kappa\lambda} g_{\nu\mu}\right] \\
& + g_{\eta\sigma}\left[\Gamma^{\eta}_{\kappa\lambda}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\eta}_{\mu\lambda}\Gamma^{\sigma}_{\nu\kappa}\right] + \\
& \frac{1}{2}\left[\partial_{\nu\kappa} g_{\lambda\mu} - \partial_{\nu\lambda} g_{\kappa\mu} - \partial_{\mu\kappa} g_{\lambda\nu} + \partial_{\mu\lambda} g_{\kappa\nu}\right] \\
& + g_{\eta\sigma}\left[\Gamma^{\eta}_{\mu\lambda}\Gamma^{\sigma}_{\kappa\nu} - \Gamma^{\eta}_{\nu\lambda}\Gamma^{\sigma}_{\kappa\mu}\right]
\end{aligned}$$

WE HAVE CHOOSEN TORSION LESS METRIC $\longrightarrow \Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$

(d) WE HAVE :

$$R_{\lambda\mu[\nu\kappa;\tau]} = \nabla_{[\tau|} R_{\lambda\mu|\nu\kappa]} =$$

JUMP IN PERMUTED INDICES.

$$\nabla_\tau R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\tau} + \nabla_\kappa R_{\lambda\mu\tau\nu}$$
$$- \nabla_\tau R_{\lambda\mu\kappa\nu} - \nabla_\nu R_{\lambda\mu\tau\kappa} - \nabla_\kappa R_{\lambda\mu\tau\kappa} = 0 ?$$

BUT $R_{\lambda\mu\kappa\nu} = -R_{\lambda\mu\nu\kappa}$

SO: $2\left[\nabla_\tau R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\tau} + \nabla_\kappa R_{\lambda\mu\tau\nu}\right] = 0$

WE CAN GO NUTS AND COMPUTE EVERYTHING BY BRUTE FORCE (IF YOU SHOW ME YOU DID IT, I WILL INVITE YOU TO A KANELBULLE) OR WE CAN BE ELEGANT; FOR THIS, WE CAN ALWAYS MOVE TO LOCAL INERTIAL FRAME WHERE $\Gamma^\alpha_{\beta\delta} = 0$, BUT NOT ITS DERIVATIVES.

HENCE:

$$\nabla_\tau R_{\lambda\mu\nu\kappa} \Rightarrow \partial_\tau R_{\lambda\mu\nu\kappa} = \partial_\tau \left(\text{SECOND } \partial \text{ OF } g_{\mu\nu}\right)$$

$$\partial_\tau R_{\lambda\mu\nu\kappa} = \frac{1}{2}\partial_\tau\left[\partial_{\kappa\mu} g_{\lambda\nu} - \partial_{\kappa\lambda} g_{\mu\nu} - \partial_{\nu\mu} g_{\lambda\kappa} + \partial_{\nu\lambda} g_{\mu\kappa}\right]$$

$$\partial_\kappa R_{\lambda\mu\tau\nu} = \frac{1}{2}\partial_\kappa\left[\partial_{\nu\mu} g_{\lambda\tau} - \partial_{\nu\lambda} g_{\mu\tau} - \partial_{\tau\mu} g_{\lambda\nu} + \partial_{\tau\lambda} g_{\mu\nu}\right]$$

$$\partial_\nu R_{\lambda\mu\kappa\tau} = \frac{1}{2}\partial_\nu\left[\partial_{\tau\mu} g_{\lambda\kappa} - \partial_{\tau\lambda} g_{\mu\kappa} - \partial_{\kappa\mu} g_{\lambda\tau} + \partial_{\kappa\lambda} g_{\mu\tau}\right]$$

✓

(2) WE HAVE TO USE PREVIOUS RELATION TO PROVE:

$$\nabla^a G_{ab} = 0 = \nabla^a\left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 0;$$

$$\left[\nabla_\tau R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\tau} + \nabla_\kappa R_{\lambda\mu\tau\nu}\right] = 0 \qquad \text{RECALL } \nabla_a g_{\mu\lambda} = 0$$

$$g_{\alpha\lambda}\left(\nabla_\tau R^\alpha{}_{\mu\nu\kappa} + \nabla_\nu R^\alpha{}_{\mu\kappa\tau} + \nabla_\kappa R^\alpha{}_{\mu\tau\nu}\right) = 0;$$

SAY $\alpha = \nu \Rightarrow$ CONTRACT:

$$g_{\alpha\lambda}\left(\nabla_\tau R_{\mu\kappa} - \nabla_\kappa R_{\mu\tau} + \nabla_\nu R^\nu{}_{\mu\kappa\tau}\right) = 0$$

CONTRACT AGAIN $g^{\mu\kappa}(\ \ ) =$

$$g_{\alpha\lambda}\left(\nabla_\tau R - \nabla^\mu R_{\mu\tau} - \nabla_\nu g^{\mu\kappa} R^\nu{}_{\mu\tau\kappa}\right) = 0$$

$$\left(\nabla_\tau R - \nabla^\mu R_{\mu\tau} - \nabla_\nu R^\nu{}_\tau\right) = 0$$

$$\left(\frac{1}{2}\nabla_\tau R - \nabla_\mu R^\mu{}_\tau\right) = 0 \Rightarrow \left(\frac{1}{2}\delta^\mu_\tau \nabla_\mu R - \nabla_\mu R^\mu{}_\tau\right) = 0$$

$g^{\tau\alpha}$

$$= \nabla_\mu \left(\frac{1}{2} g^{\mu\alpha} R - R^{\mu\alpha} \right) \Rightarrow \nabla^a \left(\underbrace{R_{ab} - \frac{1}{2} g_{ab} R}_{G_{ab} \ \checkmark} \right) = 0$$

NG3 RIEMMAN TENSOR PROPERTIES AND SYMS

TO STUDY THIS PROBLEM BE CAN USE PETROV NOTATION;

WE CAN THINK OF $R_{\alpha\mu\nu\kappa}$ AS $R_{(A)(B)}$, SO A MATRIX.

THEN WE KNOW THAT $R_{\alpha\mu\nu\kappa} = R_{\nu\kappa\alpha\mu} \Rightarrow R_{AB} = R_{BA}$.

BUT EACH INDEX A, B BEHAVES ANTISYM AS $R_{\alpha\mu\cdot\cdot} = -R_{\mu\alpha\cdots}$

SO EACH INDEX "A" TAKES A # OF INDEPENDANT VALUES EQUAL

TO THE NUMBER OF INDEPENDANT ENTRIES OF AN ANTISYM MATRIX.

SO FROM $R_{AB} = R_{BA} \longrightarrow \dfrac{N(N+1)}{2}$

BUT THAT N IS NOT THE DIMENSION, AS IT COUNTS FOR

TWO INDICES. THESE TWO INDICES ARE ANTISYM SO

$$N = \frac{D(D-1)}{2}$$

ON TOP OF THAT, WE HAVE BIANCHI, WHICH WILL IMPOSE

EXTRA CONSTRAINTS. THIS IS GIVEN BY $\begin{pmatrix} D+r-1 \\ r \end{pmatrix}$ COMPLETE SYM PROPERTIES.

WITH r THE RANK OF $R_{\alpha\mu\nu\kappa}\ldots(4)$.

SO WE HAVE THAT THE NUMBER OF INDEPENDENT COMPONENTS IS:

$$\#_{IND} = \frac{N(N+1)}{2} - \binom{D+r-1}{r} =$$

$$= \frac{\frac{D(D-1)}{2} \cdot \frac{D(D-1)}{2} + 1}{8} - \frac{(D+r-1)(D+r-2)\dots D}{r!}$$

$$\Rightarrow \frac{1}{12} D^2 (D^2 - 1)$$

OBSERVE:

D	1	2	3	4	....
#	0	1	6	20	

073 SOME BIANCHI IDENTITIES.

FOR $A_{\mu\nu} = -A_{\nu\mu}$ PROVE $\nabla_{[\mu}A_{\nu\lambda]} = \partial_{[\mu}A_{\nu\lambda]}$

RECALL THAT

$$\nabla_\mu A_{\nu\lambda} = \partial_\mu A_{\nu\lambda} \overset{+\text{UP}}{\underset{-\text{DOWN}}{-}} \Gamma^{\alpha}_{\mu\nu} A_{\alpha\lambda} - \Gamma^{\alpha}_{\mu\lambda} A_{\nu\alpha}$$

THEN WE HAVE:

$$\nabla_{[\mu}A_{\nu\lambda]} = \nabla_\mu A_{\nu\lambda} + \nabla_\nu A_{\lambda\mu} + \nabla_\lambda A_{\mu\nu}$$
$$- \nabla_\mu A_{\lambda\nu} - \nabla_\nu A_{\mu\lambda} - \nabla_\lambda A_{\nu\mu}$$

BUT! AS $A_{\mu\nu} = -A_{\nu\mu}$, WE GET TWICE EACH PIECE.

SO:

$$\nabla_\nu A_{\lambda\mu} = \partial_\nu A_{\lambda\mu} - \Gamma^{\alpha}_{\nu\lambda} A_{\alpha\mu} - \Gamma^{\alpha}_{\nu\mu} A_{\lambda\alpha}$$
$$\nabla_\lambda A_{\mu\nu} = \partial_\lambda A_{\mu\nu} - \Gamma^{\alpha}_{\lambda\mu} A_{\alpha\nu} - \Gamma^{\alpha}_{\lambda\nu} A_{\mu\alpha}$$

BUT $A_{\mu\nu} = -A_{\nu\mu}$, AND $\Gamma_{ij} = \Gamma_{ji}$ (TORSIONLESS). SO GREEN DOTS CANCEL AND WE ARE LEFT WITH $\partial_i A_{jk}$. BUT $\partial_i A_{jk} = -\partial_i A_{kj}$. OPEN UP THE 2 FACTOR TO SEE $\nabla_{[\mu}A_{\nu\lambda]} = \partial_{[\mu}A_{\nu\lambda]}$ ✓

081 SYMMETRIES OF RIEMANN TENSOR

WE HAVE A SIMPLE RIEMANN TENSOR AS:

$$R_{\mu\nu\sigma\kappa} = \lambda \left(g_{\mu\sigma} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\sigma} \right)$$

① CHECK SYMMS:

- IF $\sigma \iff \mu$ AND $\kappa \iff \nu$

  RHS $= \lambda \left(g_{\sigma\mu} g_{\kappa\nu} - g_{\sigma\nu} g_{\kappa\mu} \right)$ IF $g_{\mu\nu} = g_{\nu\mu} \implies$

  RHS $= R_{\sigma\kappa\mu\nu}$ ✓

- IF $\nu \iff \mu$

  RHS $= \lambda \left(g_{\nu\sigma} g_{\mu\kappa} - g_{\nu\kappa} g_{\mu\sigma} \right) \implies -R_{\nu\mu\sigma\kappa}$ ✓

- $R_{\mu[\nu\rho\sigma]} = 0$ (YOU CAN CHECK IT :))

② $R_{\nu\kappa}$? MULTIPLY $g^{\sigma\mu}$ BOTH SIDES:

$$R^{\sigma}{}_{\nu\sigma\kappa} = \lambda \left(\underbrace{g^{\sigma\mu} g_{\mu\sigma}}_{\text{DIM}} g_{\nu\kappa} - \underbrace{g^{\sigma\mu} g_{\mu\kappa}}_{\delta^{\sigma}_{\kappa}} g_{\nu\sigma} \right) =$$

$\Rightarrow$

$$R_{\nu\kappa} = \lambda(D-1)\, g_{\nu\kappa} :$$

$\Rightarrow$ MULTIPLY BOTH SIDES $g^{\nu\kappa}$

$$\boxed{R = \lambda(D-1)D}$$

③ $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

$$= \lambda(D-1)\, g_{ab} - \frac{1}{2} g_{ab} \lambda (D-1) D$$

$$= \lambda g_{ab} \left(\frac{2(D-1)}{2} - \frac{(D-1)D}{2} \right)$$

$$= -\lambda g_{ab} \left(\frac{(D-2)(D-1)}{2} \right)$$

RECALL THAT BIANCHI TRANSLATES IN COVARIANT CONSERVATION.

SO $\nabla^a G_{ab}$ = NUMBER. $\nabla^a g_{ab} = 0$ ✓

082 CURVATURE IN 2 DIM.

FOR $D=2 \rightarrow \exists$ 1 INDEPENDANT COMPONENT OF $R_{\mu\nu\sigma\rho} \rightarrow R_{1212}$.

① WE HAVE TO SHOW :

$$R = \frac{2}{g_{11}g_{22} - g_{12}g_{21}} R_{1212}.$$

LET'S START FROM $R_{\mu\nu\sigma\rho}$, WHAT IS THIS? HOW DOES THIS RELATED TO $R_{\mu\nu}$?

RECALL THAT $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = g^{\sigma\rho} R_{\rho\mu\sigma\nu}$

BUT RECALL THAT!

$$R_{\alpha(\mu\nu\kappa)} = 0$$

$$R_{\alpha\mu\nu\kappa} = R_{\nu\kappa\alpha\mu} \quad \text{SYM BY PAIRS}$$

$$= -R_{\mu\alpha\nu\kappa} = -R_{\alpha\mu\kappa\nu} \quad \text{ANTISYM BY SINGLE.}$$

SO $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} = R^{1}_{\mu 1\nu} + R^{2}_{\mu 2\nu} =$

$$g^{11} R_{1\mu 1\nu} + g^{12} R_{2\mu 1\nu} + g^{22} R_{2\mu 2\nu} + g^{21} R_{1\mu 2\nu} =$$

$$g^{11}R_{1\mu1\nu} + g^{12}R_{2\mu1\nu} + g^{22}R_{2\mu2\nu} + g^{21}R_{1\mu2\nu}$$

THIS MEANS, THAT GIVEN SYMS OF THE SYSTEM :

$$\mu=1, \nu=1 \Rightarrow R_{11} = g^{11}\cancel{R_{1111}} + g^{12}\cancel{R_{2111}} + g^{22}R_{2121} + g^{21}\cancel{R_{1121}}$$

$$\mu=1, \nu=2 \Rightarrow R_{12} = g^{11}\cancel{R_{1112}} + g^{12}R_{2112} + g^{22}\cancel{R_{2122}} + g^{21}R_{1122}$$

$$\mu=2, \nu=1 \Rightarrow R_{21} = g^{11}\cancel{R_{1211}} + g^{12}R_{2211} + g^{22}\cancel{R_{2221}} + g^{21}R_{1221}$$

$$\mu=2, \nu=2 \Rightarrow R_{22} = g^{11}R_{1212} + g^{12}\cancel{R_{2212}} + g^{22}\cancel{R_{2222}} + g^{21}\cancel{R_{1222}}$$

WELL, WHY HAVE DONE THIS ? EASY, JUST B.C:

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22}.$$

$$= g^{11}\left(g^{22}R_{2121}\right) + g^{12}\left(g^{12}R_{2112} + g^{21}R_{1122}\right) +$$
$$g^{21}\left(g^{12}R_{2211} + g^{21}R_{1221}\right) + g^{22}\left(g^{11}R_{1212}\right)$$

APPLY SYMS OF $R_{\mu\nu\rho\sigma} \Rightarrow$

$$R = g^{11}\left(g^{22}R_{1212}\right) + g^{12}\left(-g^{12}R_{1212} + g^{21}\left(-R_{1212} - R_{1221}\right)\right)$$
$$g^{21}\left(g^{12}\left(-R_{2121} - R_{2112}\right) - g^{21}R_{1212}\right) + g^{22}\left(g^{11}R_{1212}\right)$$

$$= g^{11}\left(g^{22} R_{1212}\right) + g^{12}\left(-g^{12} R_{1212} + g^{21}\left(-\cancel{R_{1212}} + \cancel{R_{1212}}\right)\right) +$$

$$g^{21}\left(g^{12}\left(-\cancel{R_{1212}} + \cancel{R_{1212}}\right) - g^{21} R_{1212}\right) + g^{22}\left(g^{11} R_{1212}\right)$$

$$= \left(g^{11}g^{22} - g^{12}g^{12} - g^{21}g^{21} + g^{22}g^{11}\right) R_{1212} = R$$

$$\Rightarrow 2\left(g^{11}g^{22} - g^{12}g^{21}\right) R_{1212} = R$$

NOW, USE THE 2D METRIC RELATION TO SEE:

$$g^{11} = \frac{g_{22}}{|g|}, \quad g^{22} = \frac{g_{11}}{|g|},$$

$$g^{12} = -\frac{1}{|g|} g_{21}, \quad g^{21} = -\frac{g_{12}}{|g|}.$$

SO $$R = 2\left(\frac{1}{|g|^2}\overbrace{\left(g_{22}g_{11} - g_{12}g_{21}\right)}^{|g|}\right) R_{1212}$$

$$\boxed{= \frac{2}{|g|} R_{1212}}$$

QED.

(2) TO COMPUTE R OF $ds^2 = dx^2 + e^{2y} dy^2$, RECALL THAT

$\exists$ 1 RIEMMAN.

$$R_{xyxy} = g_{xx} R^x{}_{yxy} = 1 \cdot \left(\partial_x \Gamma^x_{yy} - \partial_y \Gamma^x_{yx} + \Gamma^x_{x\alpha} \Gamma^\alpha_{yx} - \Gamma^x_{y\alpha} \Gamma^\alpha_{yy} \right)$$

= AFTER COMPUTING CHRISTOFFELS =

$$= \partial_x \Gamma^x_{yy} - \Gamma^x_{xy} \Gamma^y_{xx} = - e^{2x} :$$

$$R^x{}_{yxy} = - e^{2x} = R_{xyxy}$$

BUT USE WHAT YOU GOT BEFORE TO SEE THAT:

$$R = \frac{2}{|1 \cdot e^{2x} - 0 \cdot 0|} - e^{2x} = \boxed{-2 = R}$$

083 CURVATURE IN 2+1 DIM.

WE KNOW THAT THE NUMBER OF INDEPENDANT COMPONENTS IS

$\#_{IND} = \frac{1}{12} D^2 (D^2 - 1)$ AS WE GOT IN PREVIOUS EXERCISES. FOR $D=3$

$\Rightarrow \# = 6$. THOSE ARE:

$$\{ R_{0101}, R_{0102}, R_{0112}, R_{0202}, R_{0212}, R_{1212} \}$$

AS THE RICCI IS SYM WE HAVE $\#_{IND} = \frac{D(D+1)}{2} = 6$, WHICH

ARE: $\{ R_{00}, R_{01}, R_{02}, R_{11}, R_{12}, R_{22} \}$.

RECALL THAT $R_{ij} = g^{\alpha\beta} R_{\alpha i \beta j} \dots$

SO WE CAN EXPRESS EACH $R_{\lambda\nu\rho\kappa}$ IN TERMS OF

$R_{\nu\kappa}$ AS:

$R_{1010} = \frac{1}{g_{11}} R_{00} + \dots$ 6 UNKNOWN $R_{\mu\nu\sigma\kappa}$ FOR 6 FIXED $R_{\nu\kappa}$.

FINALLY CONSIDER $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \leftarrow$ BECAUSE VACUA

CONTRACT $\Rightarrow \left(1 - \frac{D}{2}\right) R = 0 \rightarrow 1 - \frac{D}{2} \neq 0 \Rightarrow \boxed{R = 0}$

WHICH IMPLIES $R_{\mu\nu} = 0$ $\rightarrow$ BUT IF $R_{\mu\nu} = 0$ $\rightarrow$ GIVEN THE PREVIOUS SYSTEM OF EQS, WE SEE $R_{\alpha\mu\nu\kappa} = 0$ ✓

084 CURVATURE IN 3+1 DIMENSIONS

THIS GONNA BE OUR FIRST HEAVY COMPUTATIONAL PROBLEM:

THE LINE INVARIANT IS:

$$ds^2 = -dt^2 + dz^2 + f^2(z)\,(dr^2 + r^2 d\theta^2)$$

SOME SORT OF CYLINDRICAL COORDINATES. THE METRIC:

$$g_{\mu\nu} = \mathrm{Diag}\{-1, 1, f(z), f(z) r^2\}$$

NOW, WE SET THE MACHINERY UP:

1. $\Gamma^{\sigma}_{\mu\nu}$? OBSERVE THAT "t" AND "z" ARE "EMPTY". JUST A BARE FLAT COORDINATE. ON TOP OF THAT, NOTHING DEPENDS ON "t", SO WE CAN ARGUE THAT $\nexists\ \Gamma$ WITH t-index.

   FOR "z", OBSERVE THAT r AND θ ARE COORDINATES WITH z DEPENDENCE (DUE TO $f(z)$), THIS MEANS THAT THINGS OF THE FORM $\partial_z g_{rr}$ AND $\partial_z g_{\theta\theta}$ WILL CONTRIBUTE TO Γ SO:

$$\Gamma^z_{zr} \propto \partial_z g_{rz} + \partial_r g_{zz} \Rightarrow 0$$, BUT:

$$\Gamma^z_{rr} = \frac{1}{2} g^{z\overset{=z}{\sigma}} \left(\cancel{\partial_r} g_{\sigma r} + \cancel{\partial_r} g_{r\sigma} - \partial_\sigma g_{rr} \right) =$$

$$= -\frac{1}{2} g^{zz} \partial_z g_{rr} = -f(z) f'(z)$$

SIMILARLY:

$$\Gamma^z_{\theta\theta} = -\frac{1}{2} g^{zz} \partial_z g_{\theta\theta} = -r^2 f'(z) f(z)$$

NOW, WE DIVE INTO DANGEROUS ZONE! CHRISTOFFELS W/ $\Gamma^r_{\alpha\beta}$ CANNOT BE Γ^r_{zz}, BECAUSE:

$$\Gamma^r_{zz} \propto \cancel{\partial_z g_{zr}} + \cancel{\partial_z g_{rz}} - \underbrace{\cancel{\partial_r} \overset{1}{g_{zz}}}_{0} = 0$$

SAME WITH "t", BUT THEY CAN BE $\Gamma^r_{rz} = \Gamma^r_{zr}$ SO:

$$\Gamma^r_{zr} = \frac{1}{2} g^{r\overset{=r}{\sigma}} \left(\cancel{\partial_r g_{\sigma z}} + \partial_z g_{r\sigma} - \cancel{\partial_\sigma g_{zr}} \right) =$$

$$= \frac{1}{2} g^{rr} \partial_z g_{rr} = \frac{f'}{f}$$

ALSO, DO NOT FORGET $\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} \partial_r g_{\theta\theta} = -r$.

WHEN $\Gamma^{\theta}_{\alpha\beta}$, WE HAVE SIMILAR CASE, AS BEFORE WITH:

$$\Gamma^{\theta}_{z\theta} = \frac{1}{2} g^{\theta\theta} \partial_z g_{\theta\theta} = f' f = \Gamma^{\theta}_{\theta z}$$

ALSO:

$$\Gamma^{\theta}_{rr} = \frac{1}{2} g^{\theta\theta} \partial_r g_{rr} = \frac{1}{r}$$

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2} g^{\theta\theta} \partial_r g_{\theta\theta} = \frac{1}{r}$$

SO WE HAVE $\neq 0$:

- $\Gamma^{z}_{rr} = -f(z) f'(z)$
- $\Gamma^{z}_{\theta\theta} = -r^2 f f'$
- $\Gamma^{r}_{zr} = \Gamma^{r}_{rz} = f'/f$
- $\Gamma^{r}_{\theta\theta} = -r$
- $\Gamma^{\theta}_{z\theta} = \Gamma^{\theta}_{\theta z} = f'/f$
- $\Gamma^{\theta}_{r\theta} = 1/r$

② TO COMPUTE THE RIEMANN WE HAVE TO APPLY SAME STORY:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

OBSERVE THAT IF $\rho = t$, WE HAVE NO Γ WITH t INDEX,

SO WE CAN AVOID COMPUTING THAT:

WHAT ABOUT $R^z{}_{...}$? WE KNOW THAT NO t INDEX WOULD POP UP.

WHAT IF $R^z{}_{z\mu\nu}$?

$$R^z{}_{z\mu\nu} = \partial_\mu \Gamma^z_{\nu z} - \partial_\nu \Gamma^z_{\mu z} + \Gamma^z_{\mu\lambda}\Gamma^\lambda_{\nu z} - \Gamma^z_{\nu\lambda}\Gamma^\lambda_{\mu z}$$

RECALL THAT $R^\rho{}_{\sigma\mu\nu} = - R^\rho{}_{\sigma\nu\mu}$!

THE ONLY $\Gamma^z_{\nu z}$ WE HAVE IS Γ^z_{rz} ! WHAT IF $R^z{}_{zrr}$ =

$$= \partial_r \Gamma^z_{\nu z} - \partial_r \Gamma^z_{rz} + \Gamma^z_{r\lambda}\Gamma^\lambda_{rz} - \Gamma^z_{r\lambda}\Gamma^\lambda_{rz} = 0.$$

AND $R^z{}_{zrz}$?

$$= \partial_r \underset{\nexists}{\cancel{\Gamma^z_{zz}}} - \partial_z \overset{\nexists}{\underset{0}{\cancel{\Gamma^z_{rz}}}} + \Gamma^z_{r\lambda} \overset{\nexists}{\cancel{\Gamma^\lambda_{zz}}} - \underset{0}{\cancel{\Gamma^z_{z\lambda}}} \underset{\nexists}{\Gamma^\lambda_{rz}}$$

APPLYING THE SYMMETRICAL PROPERTIES, ALL POSSIBLE COMBINATIONS ARE 0. WHAT ABOUT $zrzr$ AND $z\theta z\theta$ COMBINATIONS?

$$R^{z}{}_{rzr} = \partial_z \Gamma^{z}_{rr} - \partial_r \Gamma^{z}_{zr} + \Gamma^{z}_{z\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{z}_{r\lambda}\Gamma^{\lambda}_{zr}$$

$$= \partial_z(-ff') - \emptyset + \emptyset - \Gamma^{z}_{rr}\Gamma^{r}_{zr}$$

$$= -f''f - f'^2 + \frac{ff'^2}{f} = -f''f$$

USING SYMMETRIES $R^{z}{}_{rzr} = -R^{z}{}_{rrz}$ ✓

AS $\Gamma^{z}_{\theta\theta} = r^2\Gamma^{z}_{rr} \Longrightarrow$

$R^{z}{}_{\theta z\theta} = r^2 R^{z}{}_{rzr} \Longrightarrow$ COMPUTE IF YOU DO NOT BELIEVE!

OBSERVE THAT $g_{\gamma\rho}R^{\rho}{}_{\sigma\mu\nu} = R_{\gamma\sigma\mu\nu} \Rightarrow$ SO YOU CAN APPLY MORE SYM!

EJ!

$$R^{z}{}_{rzr} \Rightarrow g^{z\gamma}R_{\gamma rzr} = -g^{z\gamma}R_{r\gamma zr} =$$

$$= -g^{z\gamma}g_{r\beta}R^{\beta}{}_{\gamma zr}$$

AS METRIC IS SYM $\longrightarrow$ $= -g^{zz}g_{rr}R^{r}{}_{zzr}$

$$= -1.\, f(z)^2 R^{r}{}_{zzr} \Rightarrow$$

$$R^r{}_{zzr} = -\frac{1}{f^2} R^z{}_{rzr} = +\frac{f''}{f} ;$$

APPLY SAME ARG AS BEFORE TO SEE $R^r{}_{zzr} = -R^r{}_{zrz}$.

SAME STORY TO SEE $R^\theta{}_{z z\theta} = f''/f$.

WHAT ABOUT

$$\begin{aligned} R^r{}_{\theta\theta r} &= \partial_\theta \Gamma^r_{\theta r} - \partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{\theta\lambda}\Gamma^\lambda_{r\theta} - \Gamma^r_{r\lambda}\Gamma^\lambda_{\theta\theta} \\ &= 1 + (0 + (-1)) - \Gamma^r_{rz}\Gamma^z_{\theta\theta} \\ &= -r^2 ff' \cdot \frac{f'}{f} = -r^2 f'^2 \end{aligned}$$

LASTLY, OBSERVE THAT:

$$R^r{}_{\theta\theta r} = g^{rr} R_{r\theta\theta r} = -g^{rr} R_{\theta r\theta r} = -g^{rr} g_{\theta\theta} R^\theta{}_{r\theta r}$$

$$\begin{aligned} \text{SO} \quad R^\theta{}_{r\theta r} &= g_{rr}\, g^{\theta\theta} R^r{}_{\theta\theta r} \\ &= f^2 \frac{1}{r^2 f^2} (-r^2 f'^2) = -f'^2 \end{aligned}$$

ALBRICIAS! WE HAVE EVERYTHING! AT LAST! TIME TO FIND THE RICCI AND SCALAR.

$\Longrightarrow$

③ RECALL THAT $R_{\mu\nu} = R^{\rho}_{\ \mu\rho\nu}$, SO WE ARE AFTER THOSE FORMS.

DUE TO SYM OF METRIC, $R_{\mu\mu}$. WE HAVE THE FOLLOWING THEN!

$R^{t}_{\ \alpha t \alpha} = 0$ WITH $\alpha = \{z, r, \theta\}$

$R^{z}_{\ rzr} = -f''f$; $R^{z}_{\ \theta z \theta} = -r^2 f'' f$

$R^{r}_{\ zrz} = \dfrac{-f''}{f}$; $R^{r}_{\ \theta r \theta} = -r^2 f'^2$

$R^{\theta}_{\ z\theta z} = \dfrac{-f''}{f}$; $R^{\theta}_{\ r\theta r} = -f'^2$

SO: $R_{tt} = 0$

$$R_{zz} = R^{\delta}_{\ z\delta z} = R^{r}_{\ zrz} + R^{\theta}_{\ z\theta z} = -\frac{2f''}{f}$$

$$R_{rr} = R^{\delta}_{\ r\delta r} = R^{z}_{\ rzr} + R^{\theta}_{\ r\theta r} = -f'^2 - f''f$$

$$R_{\theta\theta} = \text{''} = -r^2\left(ff'' + f'^2\right)$$

AND THE SCALAR!

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{zz} R_{zz} + \dots =$$

$$= -\frac{2f''}{f} + \frac{1}{f^2}\left(-f'^2 - f''f\right) - \frac{r^2}{r^2 f^2}\left(f''f + f'^2\right) =$$

$$= -\frac{4f''}{f} - \frac{2f'^2}{f^2}$$

✓

091 MORE RIEMANN COMPUTATIONS

WE ARE GIVEN THE FOLLOWING LINE INVARIANT:

$$ds^2 = \left(-dt^2 + dx^2 + dy^2\right) e^{2\beta z} + dz^2$$

OBSERVE THAT $\{t, x, y\}$ ENTRIES HAS THE SAME FACTOR $e^{2\beta z}$ INFRONT.

LET'S COMPUTE CHRISTOFFELS.

(1) AS $\{t, x, y\}$ HAVE SAME ENTRY OBSERVE THAT:

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\mu}\right)$$

$$\Gamma^{\mu}_{\mu\nu} = -\beta \qquad \text{IF} \quad \mu = \{x, y, t\} + \nu = \{z\} \text{ ; ELSE } = 0.$$

AS THE ONLY ARGUMENT IN $g_{\mu\nu}$ IS z, WE ALSO CAN SEE THAT:

$$\Gamma^{z}_{\mu\mu} = \frac{1}{2} g^{zz}\left(\partial_\mu g_{z\mu} + \partial_\mu g_{\mu z} - \partial_z g_{\mu\mu}\right) =$$

$$\Gamma^{z}_{tt} = -\beta e^{-2\beta z} \quad ; \quad \Gamma^{z}_{xx} = \Gamma^{z}_{yy} = \beta e^{-2\beta z}$$

② THE RIEMMAN WILL FOLLOW A SIMILAR PATTERN.

IN FACT ;

$$R^{\beta}{}_{\alpha\beta\alpha} = \partial_\beta \Gamma^{\beta}_{\alpha\alpha} - \partial_\alpha \Gamma^{\beta}_{\alpha\beta} + \Gamma^{\beta}_{\beta\delta}\Gamma^{\delta}_{\alpha\alpha} - \Gamma^{\beta}_{\alpha\delta}\Gamma^{\delta}_{\alpha\beta}$$

AS THE METRIC IS FULL DIAG , WE KNOW THAT $R_{\mu\nu} \to R_{\mu\mu}$,

SO $R^{\rho}{}_{\mu\sigma\nu} = R^{\rho}{}_{\mu\sigma\mu}$. SO:

$$\left.\begin{aligned} R^{a}{}_{iai} \,, \; i,a = \{t,x,y\} &= e^{2z\beta}\beta^2 \\ R^{z}{}_{izi} &= e^{2z\beta}\beta^2 \\ R^{a}{}_{zaz} &= -\beta^2 \end{aligned}\right] + \text{SYMMETRIES}$$

THEN THE RICCI :

$$R_{tt} = R^{x}{}_{txt} + R^{y}{}_{tyt} + R^{z}{}_{tzt} = 3e^{2z\beta}\beta^2$$

$$R_{xx} = R^{t}{}_{xtx} + \dots = -3e^{2\beta z}\beta^2$$

$$R_{yy} = \qquad = \;''$$

$$R_{zz} = R^{t}{}_{ztz} + R^{x}{}_{zxz} + \dots = -3\beta^2$$

(3) FINALLY!

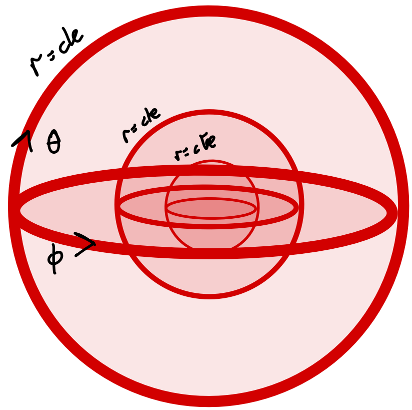
$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + \dots = -3\beta^2 - 3\beta^2 \dots = -12\beta^2!$$

N83 NORMAL VECTORS

① WE ARE GIVEN THE METRIC :

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

TO FIND THE NORMAL OF AN HYPERSURFACE (SLICE) OF THE PREVIOUS METRIC, JUST IMAGINE YOU FREEZE $r=$ constant. THIS IMPLIES : $dr = 0$! SO THE HYPER (Σ) METRIC IS:



$$ds^2_\Sigma = -dt^2 + 0 + R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2$$

WITH $R =$ cte. THIS SMELLS LIKE FIXED SPHERE + TIME RUNNING.

THE NORMAL n^μ HAS TO BE $\underbrace{\text{UNITARY}}$ AND ORTHOGONAL TO ALL TANGENT VECTORS !

+ 1 SPACELIKE
− 1 TIMELIKE

LET'S DENOTE $ds^2|_\Sigma$ AS THE INDUCED METRIC AND REFER TO IT WITH INDICES $a, b, c, d \ldots$ THE GIVEN METRIC IS THE BULK

AND IS DESCRIBED BY $\alpha, \beta, \gamma, \delta \ldots$

THE TWO EQUATIONS TO SOLVE ARE:

$$n_\alpha n^\alpha = 1 \qquad \text{AND} \qquad n_\alpha e^\alpha_a = 0$$

WHAT IS e^α_a? IS THE BASIS OF TANGENT VECTORS GIVEN BY

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$$

$\leftarrow$ coordinates in BULK $= \{t, r, \theta, \phi\}$

$\leftarrow$ COORDINATES in $\Sigma = \{t, \theta, \phi\}$

$$\text{so } e^\alpha_a = \begin{bmatrix} \frac{\partial x^0}{\partial y^0} & \frac{\partial x^0}{\partial y^1} & \cdots \\ \vdots & & \\ \frac{\partial x^3}{\partial y^0} & \cdots & \frac{\partial x^3}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

BASIS FOR TANGENT.

TO FIND THE NORMAL, ASSUME $n_\alpha = (n_0, n_1, n_2, n_3)$

SO n_i TO BE DETERMINED.

BUT $n_\alpha n^\alpha = g^{\alpha\beta} n_\alpha n_\beta = -n_0^2 + n_1^2 + r^2 n_2^2 + r^2 \sin^2\theta\, n_3^2 = 1$

AND $n_\alpha e^\alpha_a = 0 \rightarrow$

- $n_0 e^0_0 + \cancel{n_1 e^1_0} + \cancel{n_2 e^2_0} + \cancel{n_3 e^3_0} = 0 \Rightarrow n_0 \cdot (-1) = 0$
- $\cancel{n_0 e^0_1} + \cancel{n_1 e^1_1} + n_2 e^2_1 + \cancel{n_3 e^3_1} = 0 \Rightarrow n_2 \cdot 1 = 0$
- $\cancel{n_0 e^0_2} + \cancel{n_1 e^1_2} + \cancel{n_2 e^2_2} + n_3 e^3_2 = 0 \Rightarrow n_3 \cdot 1 = 0$

BUT NO TRACE OF n_1 ; SO WE FIX IT USING UNITARITY:

$n_1^2 = 1 \Rightarrow n_1 = \pm 1$ ($\pm$ IF IT POINTS OUT TO GROWING / SHRINKING SPATIAL SECTIONS.)

SO $n_\mu = n^\mu = (0, \pm 1, 0, 0)$ ✓

(2) THE CHANGE OF COORDINATES IS:

$$\left.\begin{aligned} v &= t + r \\ u &= r \end{aligned}\right] \quad t = v - u$$

SO: $dt^2 = (dv - du)^2 = dv^2 + du^2 - 2dv\,du$

$dr^2 = du^2$

NOW $d\tilde{s}^2 = -dv^2 - \cancel{du^2} + 2dv\,du + \cancel{du^2} + u^2(d\theta^2 + s^2\theta\, d\phi^2)$

IF YOU REPEAT $\eta_\mu n^\mu = 1$ AND $n_\alpha e^\alpha_a = 0$, YOU SEE THAT $\eta_\mu \neq n^\mu$.

SO THE NORMAL VECTOR n^μ IS NO LONGER η_μ. BUUUT...

$n^\mu = g^{\mu\alpha} n_\alpha$; WITH $x^\mu = (v, \theta, \phi)$

$$n^0 = g^{0\alpha} n_\alpha = g^{00} n_0 + g^{01} n_1 + g^{02} n_2 \cdots = \pm 1$$

$$n^1 = g^{1\alpha} n_\alpha = g^{10} n_0 + g^{11} n_1 + \cdots = 0$$

$$\vdots \qquad = \qquad = 0$$

$$= \qquad = 0$$

OBSERVE THAT $\check{n}^\mu = (\pm 1, 0, 0, 0)$ GOES NOW IN THE v COMPONENT!

BUT $\partial_v = \partial_t + \partial_r$ IN THE OLD COORDINATE SYSTEM.

INFINITESIMAL GENERATOR OF ISOMETRIES

N°1 KILLING VECTORS $\qquad \mathcal{L}_X g = 0 \to X \equiv$ KILLING FIELD.

MAX SYM SPACES $= \frac{n(n+1)}{2}$ KILLING

① BY JUST LOOKING AT THE METRIC WE SEE THAT:

$\partial_\mu g_{\alpha\beta} = 0 \longrightarrow \partial_t, \partial_x, \partial_y, \partial_z$ ARE KILLING. THIS CORRESPOND TO DISPLACEMENTS ON THAT FLAT METRIC.

BUT THEY ARE NOT ALL THEM!

RECALL THE KILLING EQ:

$$\nabla_{(\mu} k_{\nu)} = 0$$

$$\frac{1}{2}(\nabla_\mu k_\nu + \nabla_\nu k_\mu) = 0 \quad \Rightarrow$$

$$\frac{1}{2}(\partial_\mu k_\nu + \partial_\nu k_\mu) - \frac{1}{2}\left(\Gamma_{\mu\nu}^{\alpha} k_\alpha + \Gamma_{\nu\mu}^{\beta} k_\beta\right) = 0$$

Torsion less + flat = 0!

$$\Rightarrow \partial_\mu k_\nu = -\partial_\nu k_\mu .$$

THIS IS NOT QUITE USEFUL. WE NEED A SECOND EQ AS:

$R^{\lambda}{}_{[\sigma\rho\mu]} = 0$ WHAAAT?

RECALL THAT $\nabla_{[\rho}\nabla_{\mu]} k_\sigma = -R^{\alpha}{}_{\sigma\rho\mu} k_\alpha$

$$R^{\lambda}{}_{[\sigma\rho\mu]}k_\lambda = \nabla_\mu\nabla_\rho k_\sigma - \nabla_\rho\nabla_\mu k_\sigma + \nabla_\rho\nabla_\sigma k_\mu - \nabla_\sigma\nabla_\rho k_\mu + \nabla_\sigma\nabla_\mu k_\rho - \nabla_\mu\nabla_\sigma k_\rho$$

BUT $\nabla_i k_j = -\nabla_j k_i$

BUUUT, WE ARE ON A FLAT METRIC, SO $\Gamma = 0 \rightarrow R^{\cdots} = 0$!

SO:

$$R^{\alpha}{}_{\sigma\rho\mu}k_\alpha = -\partial_\sigma\partial_\rho k_\mu \Rightarrow$$

$$0 = \partial_\sigma\partial_\rho k_\mu$$

THE ANSATZ WE CAN USE THEN: $k_\mu = \underbrace{a_\mu}_{\text{TRANS}} + \underbrace{M_{\mu\alpha}x^\alpha}_{\text{LORENTZ}}$

BUT PLUGGING IN $\partial_\mu k_\nu + \partial_\nu k_\mu = 0$ WE SEE:

$$\partial_\mu a_\nu + \partial_\mu(M_{\nu\alpha}x^\alpha) = -\partial_\nu a_\mu - \partial_\nu(M_{\mu\alpha}x^\alpha)$$

$\Rightarrow \quad M_{\mu\nu} = -M_{\nu\mu}$. $\Leftarrow$ AS EXPECTED.

(2) THIS ONE IS WAY MORE INVOLVED, AS IT MAY EXIST SOME CURVATURE:

$$ds^2 = -r^2 dt^2 + dr^2 \implies L = -r^2 \dot{t}^2 + \dot{r}^2$$

CHRISTOFELS COME FROM EOM.

$$\dot{r} : 2\ddot{r} + 2r\dot{t}^2 = 0$$

$$\dot{t} : -2r^2\ddot{t} - 4\dot{r}r\dot{t} = 0$$

$$\ddot{t} + \frac{2}{r}\dot{t}\dot{r} = 0$$

$$\implies \boxed{\begin{aligned} \Gamma^r_{tt} &= r \\ \Gamma^t_{rt} &= \frac{1}{r} \end{aligned}}$$

ONE KILLING COULD BE ∂_t... LET SOLVE KILLING EQ.

$$\nabla_{(\mu} K_{\nu)} = 0$$

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = \partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\alpha_{\mu\nu} K_\alpha = 0$$

GIVEN THE CHRISTOFFELS;

1) $\cdot\ \partial_t K_t - \Gamma^r_{tt} K_r = 0$; $\quad \partial_t K_t = r K_r$

2) $\cdot\ \partial_r K_r = 0 \qquad\qquad \partial_r K_r = 0$

3) $\cdot\ (\partial_r K_t + \partial_t K_r) - 2\Gamma^t_{rt} K_t = 0$; $\quad (\ ) = \frac{2}{r} K_t$

2) $\partial_r k_r = 0 \rightarrow k_r = A(t) + C$

1) $\partial_t k_t = r[A(t) + C] \Rightarrow k_t = r\big(\underbrace{B(t)}_{= \int A(t)dt} + C D(r) + F\big)$

3) $\partial_r k_t + \partial_t k_r = \frac{2}{r} k_t$:

$$\big(B(t) + C D(r) + F\big) + r C \partial_r D(r) + \partial_t A(t) =$$

$$= \frac{2}{r} r \big(B(t) + C D(r) + F\big)$$

$$\Rightarrow B(t) + C D(r) + F = r C \partial_r D(r) + \partial_t A(t)$$

T WITH T AND r WITH r $\Rightarrow$

- $B(t) = \partial_t A(t)$;
- $C D(r) + F = r C \partial_r D(r) \Rightarrow$ FOR SIMPLICITY $C = 1$; $F = 0$:

ANSATZ $\Rightarrow$ $\partial_t B = \partial_t^2 A = A\,! \Rightarrow A(t) = e^{\alpha t}$, $\alpha = \pm 1$.

SO $A(t) = c_1 e^{t} + c_2 e^{-t}$

$B(t) = c_1 e^{t} - c_2 e^{-t}$

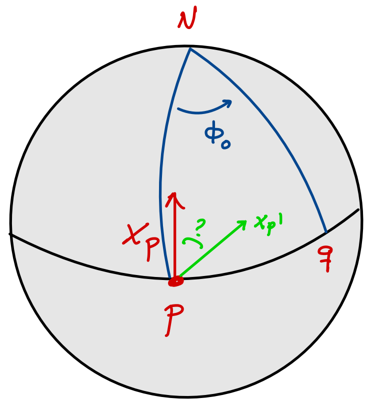
AND $D(r) = r \partial_r D(r) \Rightarrow \boxed{D(r) = c_3 r}$

IN CONCLUSION!

$$\boxed{\begin{aligned} k_r(t,r) &= c_1 e^{t} + c_2 e^{-t} \\ k_t(t,r) &= c_1 r e^{t} - c_2 r e^{-t} + c_3 r^2 \end{aligned}}$$

N61 PARALLEL TRANSPORT

WE HAVE A 2 SPHERE: $ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ AS!



BASICALLY TAKE X_p AND TRANSPORT ALONG THAT PATH. HOW WOULD IT MOVE? WELL, TIME TO APPLY PARALLEL TRANSPORT USING COVARIANT DERIVATIVE. THE COORDINATES ARE:

$$p = (\pi/2, 0)$$

$$q = (\pi/2, \phi_0)$$

$$N = (0, 0)$$

THE CHRISTOFFELS ARE: $L = \dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2 \implies$

$\theta) \quad \ddot{\theta} - 2\sin\theta\cos\theta\,\dot{\phi}^2 = 0 \quad \longrightarrow \quad \Gamma^{\theta}_{\phi\phi} = -s\theta c\theta$

$\phi) \quad \ddot{\phi} + 2ctg\theta \quad \dot{\theta}\dot{\phi} = 0 \longrightarrow \quad \Gamma^{\phi}_{\theta\phi} = ctg\theta$

SO THE PARALLEL TRANSPORT EQUATION IS:

$$u^{\alpha} \nabla_{\alpha} v^{\beta} = 0$$

WHERE: • $u^{\alpha} \equiv$ ~~NORMAL~~ TANGENT VECTOR OF THE CURVE TO TRANSPORT THROUGH.

• $v^{\beta} =$ VECTOR TO TRANSPORT. $= \bar{X}_p$

FROM P TO q

$$u^{\alpha}_{pq} = (0,1) \Rightarrow \hat{u}_{pq} = \left.\frac{u^{\alpha}}{\|u^{\alpha}\|}\right|_p = \left.\frac{1}{R\sin\theta}\right|_p (0,1) =$$

$$\left.\hat{u}_{pq}\right|_p = \frac{1}{R}(0,1). \quad (\text{B.C } \theta_p = \pi/2)$$

$v^{\beta} = (v^{\theta}, v^{\phi}) \Leftarrow \bar{X}_p$ VECTOR (WHEN WE IMPOSE SOLUTIONS)

So:

$$u^\alpha \nabla_\alpha v^\beta = 0$$

$$(u^\theta \nabla_\theta + u^\phi \nabla_\phi) v^\beta = 0$$

$$\Rightarrow \frac{1}{R} \nabla_\phi v^\beta = 0$$

$$\nabla_\phi v^\beta = (\nabla_\phi v^\theta, \nabla_\phi v^\phi)$$

$$\nabla_\phi v^\theta = \partial_\phi v^\theta + \overset{\nexists\Gamma}{\cancel{\Gamma^\theta_{\phi\theta}}} v^\theta + \Gamma^\theta_{\phi\phi} v^\phi = 0$$

$$\nabla_\phi v^\phi = \partial_\phi v^\phi + \Gamma^\phi_{\phi\theta} v^\theta + \underset{\exists\Gamma}{\cancel{\Gamma^\phi_{\phi\phi}}} v^\theta = 0$$

$$\partial_\phi v^\theta = -\Gamma^\theta_{\phi\phi} v^\phi \rightarrow \partial_\phi v^\theta = S\theta_{pq} C\theta_{pq} v^\phi$$

$$\partial_\phi v^\phi = -\Gamma^\phi_{\phi\theta} v^\theta \rightarrow \partial_\phi v^\phi = -\frac{\cos\theta_{pq}}{\sin\theta_{pq}} v^\theta$$

DERIVE ONCE RESPECT TO ϕ FIRST EQ AND PUT SECOND IN.

$$\partial^2_\phi v^\theta = -\underbrace{c^2\theta_{pq}}_{\alpha^2} v^\theta \Rightarrow v^\theta = A\cos\alpha\phi + B\sin\alpha\phi$$

DO OPOSITE :

$$\partial^2_\phi v^\phi = -c^2\theta_{pq} v^\phi \Rightarrow v^\phi = C\cos\alpha\phi + D\sin\alpha\phi$$

WE NOW HAVE TO FIX THOSE A, B, C... WE HAVE TWO SETS OF EQ'S TO FIX:

1) • $\nabla_\phi v^\beta \big|_{\phi=0} = 0$

2 • $v^\beta(\theta = \theta_0, \phi = \phi_0) = X_p^\beta$

2) $v^\theta(\phi = 0) = A \quad 1 = X_p^\theta$

$v^\phi(\phi = 0) = C \quad 1 = X_p^\phi$

APPLY NOW 1)

• $\partial_\phi v^\theta = S\theta_{pq}\, C\theta_{pq}\, v^\phi$

• $\partial_\phi v^\phi = -\dfrac{\cos\theta_{pq}}{\sin\theta_{pq}}\, v^\theta$

$\Big|\, \phi = 0$

• $-X_p^\theta\, \alpha\, \cancel{\sin\alpha\phi} + \alpha B \underset{1}{\cos\alpha\phi} = S\theta C\theta \left(X_p^\phi \cos\alpha\phi + D\cancel{\sin\alpha\phi} \right)$

• $-X_p^\phi\, \alpha\, \cancel{\sin\alpha\phi} + \alpha D \underset{1}{\cos\alpha\phi} = -\dfrac{C\theta}{S\theta} \left(X_p^\theta \cos\phi + B\cancel{\sin\alpha\phi} \right)$

$B = \dfrac{S\theta\, C\theta}{\alpha}\, X_p^\phi$

$D = \dfrac{-C\theta}{\alpha S\theta}\, X_p^\theta$

FINALLY:

$$V^\theta = X_p^\theta \cos\alpha\phi + \frac{s\theta c\theta}{\alpha} X_p^\phi \sin\alpha\phi$$
$$= X_p^\theta \cos\alpha\phi + s\theta \, X_p^\phi \sin\alpha\phi$$

SO WE HAVE THAT:

$$V^\phi = X_p^\phi \cos\alpha\phi - \frac{X_p^\theta}{\sin\theta} \sin\alpha\phi$$
$$V^\theta = X_p^\theta \cos\alpha\phi + s\theta \, X_p^\phi \sin\alpha\phi$$

IN THE CASE WE WANT TO STUDY:

$\theta_0 = \pi/2$

$p = (\pi/2, 0)$

$q = (\pi/2, \phi_0)$

$$X_q^\theta = X_p^\theta \cos(\alpha.\phi_0) + s\,\theta_{p,q} \, X_p^\phi \sin(\alpha.\phi_0)$$
$$X_q^\phi = X_p^\phi \cos(\alpha\phi_0) - \frac{X_p^\theta}{\sin\theta_0} \sin(\alpha\phi_0)$$

THE PREVIOUS EXPRESSION WOULD GIVE YOU HOW THE INITIAL VECTOR WOULD TRANSFORM AFTER DISPLACEMENT. YOU WOULD HAVE TO APPLY THIS 2 MORE TIMES TO SEE THAT THE VECTOR TILTS AN ANGLE OF ϕ_0 RESPECT TO INITIAL POSITION.

072 CONFORMAL TRANSFORMATION

A CONFORMAL TRANSFORMATION IS GIVEN BY:

$$\tilde{g}_{\mu\nu} = \omega^2(x)\, g_{\mu\nu}.$$

WE WANT NOW TO CHECK IF $\tilde{\nabla}_\mu F^{\mu\nu} = 0$ AND

$\nabla_{[\mu} F_{\nu\lambda]} = 0$.

WE KNOW $\nabla_\delta F^{\mu\nu} = \partial_\delta F^{\mu\nu} + \Gamma^\mu_{\delta\alpha} F^{\alpha\nu} + \Gamma^\nu_{\delta\alpha} F^{\mu\alpha}$

GIVEN THE NEW CONFORMAL TRANSF, THE GEOMETRICAL PROPERTIES WILL CHANGE: THE NEW CHRISTOFELS WILL BE: (APP G CARROLL)

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + C^a_{bc}, \text{ with } C^a_{bc} = \frac{1}{\omega}\left(\delta^a_b \partial_c \omega + \delta^a_c \partial_b \omega - g_{bc} g^{ad} \partial_d \omega\right)$$

SO $\tilde{\Gamma}^\mu_{\delta\alpha} = \Gamma^\mu_{\delta\alpha} + \frac{1}{\omega}\left[\partial_\alpha \omega(x)\, \delta^\mu_\delta + \partial_\delta\, \omega(x)\, \delta^\mu_\alpha - g_{\delta\alpha} g^{\mu\lambda} \partial_\lambda \omega\right]$

AND $\tilde{\Gamma}^\nu_{\delta\alpha} = \Gamma^\nu_{\delta\alpha} + \frac{1}{\omega}\left[\partial_\alpha\, \omega(x)\, \delta^\nu_\delta + \partial_\delta\, \omega(x)^\nu_\alpha - g_{\delta\alpha} g^{\nu\lambda} \partial_\lambda \omega\right]$

STARTING FROM A GENERAL $\hat{\nabla}_\nu F^{\mu\nu}$, WE SEE:

$$\hat{\nabla}_\delta F^{\mu\nu} = \tilde{\partial}_\delta F^{\mu\nu} + \hat{\Gamma}^{\mu}_{\delta\alpha} F^{\alpha\nu} + \hat{\Gamma}^{\nu}_{\delta\alpha} F^{\mu\alpha}$$

$$= \partial_\delta F^{\mu\nu} + \Gamma^{\mu}_{\delta\alpha} F^{\alpha\nu} + \Gamma^{\nu}_{\delta\alpha} F^{\mu\alpha}$$

$$+ \left(\partial_\delta w\, \delta^{\mu}_{\alpha} + \partial_\alpha w\, \delta^{\mu}_{\delta} - g_{\delta\alpha} g^{\mu\lambda} \partial_\lambda w \right) F^{\alpha\nu} \frac{1}{w}$$

$$+ \left(\partial_\delta w\, \delta^{\nu}_{\alpha} + \partial_\alpha w\, \delta^{\nu}_{\delta} - g_{\delta\alpha} g^{\nu\lambda} \partial_\lambda w \right) F^{\mu\alpha} \frac{1}{w}$$

$$\hat{\nabla}_\delta F^{\mu\nu} = \partial_\delta F^{\mu\nu} + \Gamma^{\mu}_{\delta\alpha} F^{\alpha\nu} + \Gamma^{\nu}_{\delta\alpha} F^{\mu\alpha} +$$

$$\left(\partial_\delta w F^{\mu\nu} + \partial_\alpha w F^{\alpha\nu} \delta^{\mu}_{\delta} - \partial^{\mu} w\, F_\delta{}^{\nu} \right) \frac{1}{w}$$

$$\left(\partial_\delta w\, F^{\mu\nu} + \partial_\alpha w F^{\mu\alpha} \delta^{\nu}_{\delta} - \partial^{\nu} w\, F^{\mu}{}_{\delta} \right) \frac{1}{w}$$

BUT RECALL $F_{\mu\nu} = - F_{\nu\mu}$ AND $F_{\mu\mu} = 0$ AND $\partial^\mu v_\mu = \partial_\mu v^\mu$ ...

$$= \nabla_\delta F^{\mu\nu} + \frac{1}{w} \left[2 \partial_\delta w F^{\mu\nu} + \partial_\alpha w \left(F^{\alpha\nu} \delta^{\mu}_{\delta} + F^{\mu\alpha} \delta^{\nu}_{\delta} \right) \right.$$

$$\left. - \partial^{\mu} w\, F_\delta{}^{\nu} - \partial^{\nu} w\, F^{\mu}{}_{\delta} \right]$$

IF $\delta = \mu \Rightarrow \nabla_\mu F^{\mu\nu} + \frac{1}{w} \left[2 (\partial_\mu w) F^{\mu\nu} + (\partial_\alpha w) \left(\cancel{F^{\alpha\nu}} + \cancel{F^{\nu\alpha}} \right) \right.$

MISSING 2 SOMEWHERE $\Rightarrow$ $\left. - \partial^{\mu} w\, F_\mu{}^{\nu} - \cancel{\partial^{\nu} w\, F^{\mu}{}_{\mu}} \right]$

TRACELESS

AS FOR THE BIANCHI, WE FIND:

$$\breve{\nabla}_{[\mu}F_{\nu\lambda]} = 0 \Rightarrow \breve{\nabla}_\mu F_{\nu\lambda} + \breve{\nabla}_\nu F_{\lambda\mu} + \breve{\nabla}_\lambda F_{\mu\nu}$$
$$- \breve{\nabla}_\mu F_{\lambda\nu} - \breve{\nabla}_\nu F_{\mu\lambda} - \breve{\nabla}_\lambda F_{\nu\mu} = 0$$

BUT $F_{\mu\nu} = -F_{\nu\mu}$ SO 3 COPIES WE HAVE:

$$\breve{\nabla}_\mu F_{\nu\lambda} = \breve{\partial}_\mu F_{\nu\lambda} - \breve{\Gamma}^\alpha_{\mu\nu} F_{\alpha\lambda} - \Gamma^\alpha_{\mu\lambda} F_{\nu\alpha}$$
$$\breve{\nabla}_\nu F_{\lambda\mu} = \breve{\partial}_\nu F_{\lambda\mu} - \breve{\Gamma}^\alpha_{\nu\lambda} F_{\alpha\mu} - \breve{\Gamma}_{\nu\mu} F_{\lambda\alpha}$$
$$\breve{\nabla}_\lambda F_{\mu\nu} = \breve{\partial}_\lambda F_{\mu\nu} - \breve{\Gamma}^\alpha_{\lambda\mu} F_{\alpha\nu} - \breve{\Gamma}^\alpha_{\lambda\nu} F_{\mu\alpha}$$

SO $\breve{\nabla}_{[\mu}F_{\nu\lambda]} = \breve{\partial}_{[\mu}F_{\nu\lambda]}$

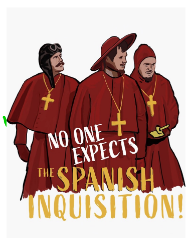
N52 COVARIANT DERIVATIVE FOR PHYSICISTS.

① DOES $\partial_\mu \omega_\nu$ TRANSFORM AS A $(0,2)$ TENSOR ?

START FROM $\quad \partial_{\mu'} \omega_{\nu'} = \dfrac{\partial}{\partial x^{\mu}} \dfrac{\partial x^{\mu}}{\partial x'^{\mu'}} \left(\dfrac{\partial x^{\nu}}{\partial x'^{\nu'}} \omega_\nu \right) \Big| =$

$$= \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \partial_\mu \left(\frac{\partial x^{\nu}}{\partial x'^{\nu'}} \omega_\nu \right) =$$

$$\underbrace{\frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial^2 x^{\nu}}{\partial x'^{\nu'} \partial x^{\mu}} \omega_\nu}_{} + \underbrace{\frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial x^{\nu}}{\partial x'^{\nu'}} \partial_\mu \omega_\nu}_{\text{THE EXPECTED TENSOR TRANS}}$$



SO IT TRANSFORM AS + EXTRA THINGS.

BASICALLY THIS IS THE PIECE THAT THE COVARIANT DERIVATIVE ACCOUNTS FOR:

② RECALL THAT :

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\lambda g_{\nu\sigma} + \partial_\nu g_{\sigma\lambda} - \partial_\sigma g_{\lambda\nu} \right)$$

IF $\Gamma^{\mu'}_{\nu'\lambda'} \propto \partial_{\lambda'} g_{\mu'\sigma'}$:

$$\partial_{\lambda'} g_{\mu'\sigma'} = \frac{\partial x^\lambda}{\partial x'^{\lambda'}} \partial_\lambda \left(\frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} g_{\mu\sigma} \right)$$

$$= \frac{\partial x^\lambda}{\partial x'^{\lambda'}} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \partial_\lambda g_{\mu\sigma}$$

$$+ \frac{\partial x^\lambda}{\partial x'^{\lambda'}} g_{\mu\sigma} \, \partial_\lambda \left(\frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \right)$$

SO $\Gamma^{\mu'}_{\nu'\lambda'} =$

$$= \frac{1}{2} \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x'^{\sigma'}}{\partial x^\sigma} g^{\mu\sigma} \Bigg[$$

$$\frac{\partial x^\lambda}{\partial x'^{\lambda'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \partial_\lambda g_{\nu\sigma} + \frac{\partial x^\lambda}{\partial x'^{\lambda'}} g_{\nu\sigma} \, \partial_\lambda \left(\frac{\partial x^\nu}{\partial x'^{\nu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \right)$$

$$+ \frac{\partial x^\nu}{\partial x'^{\nu'}} \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \frac{\partial x^\lambda}{\partial x'^{\lambda'}} \partial_\nu g_{\sigma\lambda} + \frac{\partial x^\nu}{\partial x'^{\nu'}} g_{\sigma\lambda} \, \partial_\nu \left(\frac{\partial x^\sigma}{\partial x'^{\sigma'}} \frac{\partial x^\lambda}{\partial x'^{\lambda'}} \right)$$

$$- \frac{\partial x^\sigma}{\partial x'^{\sigma'}} \frac{\partial x^\lambda}{\partial x'^{\lambda'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \partial_\sigma g_{\lambda\nu} - \frac{\partial x^\sigma}{\partial x'^{\sigma'}} g_{\lambda\nu} \, \partial_\sigma \left(\frac{\partial x^\lambda}{\partial x'^{\lambda'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \right) \Bigg]$$

GOOD OLD TENSOR PIECE

$$\Gamma^\mu_{\nu\lambda} \frac{\partial x^{\mu'}}{\partial x^\mu} \cdot \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \checkmark$$

LET'S STUDY THIS

$$+\frac{\partial x^{\lambda}}{\partial x'^{\lambda}}\, g_{\nu\sigma}\left(\overset{1}{\frac{\partial^2 x^{\nu}}{\partial x^{\lambda}\,\partial x'^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\sigma}}}+\overset{2}{\frac{\partial x^{\nu}}{\partial x'^{\nu}}\frac{\partial^2 x^{\sigma}}{\partial x^{\lambda}\,\partial x'^{\sigma}}}\right)$$

$$+\frac{\partial x^{\nu}}{\partial x'^{\nu}}\, g_{\sigma\lambda}\left(\overset{3}{\frac{\partial^2 x^{\sigma}}{\partial x'^{\sigma}\,\partial x^{\nu}}\frac{\partial x^{\lambda}}{\partial x'^{\lambda}}}+\overset{4}{\frac{\partial x^{\sigma}}{\partial x'^{\sigma}}\frac{\partial^2 x^{\lambda}}{\partial x^{\nu}\,\partial x'^{\lambda}}}\right)$$

$$-\frac{\partial x^{\sigma}}{\partial x'^{\sigma}}\, g_{\lambda\nu}\left(\overset{5}{\frac{\partial^2 x^{\lambda}}{\partial x'^{\lambda}\,\partial x^{\sigma}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}}+\overset{6}{\frac{\partial x^{\lambda}}{\partial x'^{\lambda}}\frac{\partial^2 x^{\nu}}{\partial x^{\sigma}\,\partial x'^{\nu}}}\right)$$

$$1+5 = g_{\nu\sigma}\frac{\partial x^{\lambda}}{\partial x'^{\lambda}}\frac{\partial^2 x^{\nu}}{\partial x^{\lambda}\,\partial x'^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\sigma}} \underbrace{- g_{\lambda\nu}\frac{\partial x^{\sigma}}{\partial x'^{\sigma}}\frac{\partial^2 x^{\lambda}}{\partial x'^{\lambda}\,\partial x^{\sigma}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}}_{-g_{\sigma\nu}\frac{\partial x^{\lambda}}{\partial x'^{\sigma}}\frac{\partial^2 x^{\sigma}}{\partial x'^{\lambda}\,\partial x^{\lambda}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}}$$

RELAB $\lambda \leftrightarrow \sigma$

$$= g_{\nu\sigma}\left(\frac{\partial^2 x^{\nu}}{\partial x'^{\lambda}\,\partial x'^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\sigma}} - \frac{\partial^2 x^{\sigma}}{\partial x'^{\sigma}\,\partial x'^{\lambda}}\frac{\partial x^{\nu}}{\partial x'^{\nu}}\right) = 0\,!$$

$3+6 = 0$ BY THE SAME TOKEN

$$2+4 = \frac{\cancel{\partial x^{\lambda}}}{\partial x^{\lambda}}\, g_{\nu\sigma}\,\frac{\partial x^{\nu}\ \partial^2 x^{\sigma}}{\partial x'^{\nu}\ \cancel{\partial x^{\lambda}}\,\partial x'^{\sigma}} +$$

$$\frac{\cancel{\partial x^{\nu}}}{\partial x'^{\nu}}\, g_{\sigma\lambda}\,\frac{\partial x^{\sigma}\ \partial^2 x^{\lambda}}{\partial x'^{\sigma}\ \cancel{\partial x^{\nu}}\,\partial x^{\lambda}}$$

$\Rightarrow$ MULTIPLY BY PREFACTOR

$$= \frac{1}{2} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial \cancel{x'}^{\sigma'}}{\partial \cancel{x}^{\sigma}} \overbrace{g^{\mu\sigma} g_{\nu\sigma}}^{\delta^{\mu}{}_{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\nu'}} \frac{\partial^2 \cancel{x}^{\sigma}}{\partial x'^{\lambda'} \partial \cancel{x'}^{\sigma}} +$$

$$\frac{1}{2} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \frac{\partial \cancel{x'}^{\sigma'}}{\partial \cancel{x}^{\sigma}} \underbrace{g^{\mu\sigma} g_{\sigma\lambda}}_{\delta^{\mu}{}_{\lambda}} \frac{\partial \cancel{x}^{\sigma} \; \partial^2 x^{\lambda}}{\partial x^{\nu'} \; \partial \cancel{x'}^{\sigma} \; \partial x^{\lambda'}} =$$

$$= \frac{1}{2} \frac{\partial x'^{\mu'}}{\partial x^{\mu}} \left[\frac{\partial^2 x^{\mu}}{\partial x'^{\nu'} \partial x^{\lambda'}} + \frac{\partial^2 x^{\mu}}{\partial x^{\nu'} \partial x^{\lambda'}} \right] = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\nu'} \partial x'^{\lambda'}}$$

SO WE PROVED THAT:

$$\Gamma^{\mu'}_{\nu'\lambda'} = \Gamma^{\mu}_{\nu\lambda} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{\lambda}}{\partial x'^{\lambda'}} + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\nu'} \partial x'^{\lambda'}} \checkmark$$

③ NOW THAT WE KNOW HOW Γ TRANSFORM IF WE MOVE

THE COVARIANT DERIVATIVE AS:

$$\nabla_{\mu'} w_{\nu'} = \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial^2 x^{\nu}}{\partial x'^{\nu'} \partial x^{\mu}} w_{\nu} + \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial x^{\nu}}{\partial x'^{\nu'}} \partial_{\mu} w_{\nu}$$

$$\left(\Gamma^{\alpha}_{\mu\nu} \frac{\partial x'^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x'^{\nu'}} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu'} \partial x'^{\nu'}} \right) \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} w_{\alpha}$$

$$=$$

$$= \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \partial_\mu \omega_\nu - \Gamma^{\alpha}_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \omega_\alpha$$ = GOOD TRANS

$$\cancel{\frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial^2 x^\nu}{\partial x'^{\nu'} \partial x'^\mu} \omega_\nu} - \cancel{\frac{\partial^2 x^\alpha}{\partial x^{\mu'} \partial x^{\nu'}} \omega_\alpha}$$ ✓✓

IT TRANSFORM AS WE WANT!

N53 COVARIANT DERIVATIVE FOR MATHEMATICIANS.

SO WE HAVE A MAP ∇, s.t.

$$\nabla : (X, T) \longrightarrow \nabla_X T$$

coordinate basis

(1) $$\nabla_\mu x^\nu \doteq (\nabla_{e_\mu} x)(\theta^\nu) = \left[\nabla_{e_\mu}(x^\rho e_\rho)\right]^\nu =$$

$$= \left[(\nabla_{e_\mu} x^\rho) e_\rho + x^\rho (\nabla_{e_\mu} e_\rho)\right]^\nu = \text{APPLY DEF}$$

$$= \left[(\partial_\mu x^\rho) e_\rho + x^\rho \Gamma^\alpha_{\mu\rho} e_\alpha\right]^\nu$$

BACK TO INITIAL BASIS.
$$= \partial_\mu x^\nu + \Gamma^\alpha_{\mu\nu} x^\nu$$

(2) $$\nabla_\mu \eta_\nu = (\nabla_\mu \eta)(e_\nu) = (\nabla_{e_\mu} \eta_\rho \theta^\rho)_\nu =$$

$$= \left[(\nabla_{e_\mu} \eta_\rho)\theta^\rho + \eta_\rho (\nabla_{e_\mu} \theta^\rho)\right]_\nu = \text{APPLY DEF}$$

$$= \left[(\partial_\mu \eta_\rho)\theta^\rho + \eta_\rho \underbrace{(\nabla_{e_\mu} \theta^\rho)}_{?}\right]_\nu$$

WE KNOW $\nabla_{e_\mu}(\theta^\rho e_\nu) = 0 = (\nabla_{e_\mu}\theta^\rho) e_\nu +$
$(\nabla_{e_\mu} e_\nu)\theta^\rho = (\nabla_{e_\mu}\theta^\rho) e_\nu + \Gamma^\rho_{\mu\nu}$ SO

$$= \left[(\partial_\mu \eta_\rho)\,\theta^\rho + \eta_\rho\, \underbrace{(\nabla_{e_\mu} \theta^\rho)}_{-\Gamma^\rho_{\mu\nu}} \right]_\nu$$

$$\Rightarrow \quad \partial_\mu \eta_\nu - \eta_\rho \Gamma^\rho_{\mu\nu}$$

Black holes

Schwarzchild Geometry

Consider a massive particle following a geodesic in the Schwarzchild geometry.

1. Find equations for $\dot{t}$ and $\dot{\phi}$, up to integration constants.
2. Find an expression for $\dot{r}$ using the previous results (assume $\sin\theta = 1$ and $\dot{\theta} = 0$).
3. Find values of r for which the orbits are circular.

Clocks orbiting Black holes

Suppose we have a clock in a circular orbit around a spherically symmetric object of mass M at radius R.

1. What is the time dilation of the clock as seen by a fixed observer at the same radius R ?
2. What is the time dilation of the clock as seen by a fixed observer at infinity?

Deriving the Reissner-Nordström black hole line invariant

Before we dive into computing some orbits for this type of black hole, let us derive the explicit form of its metric. We can start with the following Ansatz:

$$ds^2 = -A(t,r)dt^2 + B(t,r)dr^2 + r^2 d\Omega_2^2, \tag{1}$$

where $d\Omega_2^2$ is the usual two sphere volume and A, B some functions depending on t and r. As we may recall from the lectures, this type of black holes also have an associated charge Q. If there is a charge around, there must be an associated electric field E around, which means that we have an electric energy density in the RHS of Einstein equations. The electromagnetic stress-energy tensor reads as:

$$T_{\alpha\beta} = \left(\frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu} - g_{\beta\nu} F_{\alpha\mu} F^{\nu\mu}\right). \tag{2}$$

Recall that $F_{\mu\nu}$ is the field strength of the vector field A_μ. In order to keep things simple, let us assume the presence of only an electric field in the radial direction, which translates to:

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_r & 0 & 0 \\ -E_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Making good use the source-free Maxwell equations i.e.

$$\begin{aligned} &\nabla_\beta F^{\alpha\beta} = 0, \\ &F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0, \end{aligned} \tag{4}$$

Solve Einstein equations such that you find explicit expressions for A, B. The final result is the line invariant given in the next exercise.

Reissner-Nordström black hole

Consider the Reissner-Nordström black hole. The metric is

$$ds^2 = -\left(1 - \frac{m}{r} + \frac{Q^2}{r^2}\right)^1 dt^2 + \left(1 - \frac{m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

1. Make extremal, i.e. compute the requirement(s) to have the minimal value for the horizon.
2. What are the (obvious) conserved quantities?
3. In terms of these, what are the radii of stable and unstable orbits for massive uncharged particles?
4. Find the radii for a photon to have a circular orbit around this black hole. Choose to work with $\sin\theta = 1$.

Extremal charged black hole

Consider the extremal Reissner-Nordström (RN) black hole. Starting from the general RN solution, show the metric reduces to

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2$$

in units where $G = 1$.

1. This solution is valid for $r > M$. By defining Eddington-Finkelstein coordinates argue that this is a coordinate singularity and recover the usual black hole and white hole regions.
2. By considering the proper length of a radial curve from $r = r_0$ to $r = M$ at constant (t, θ, ϕ), show that it is infinite. This is known as the infinite throat of the extremal RN solution. Is this present in the Schwarzschild solution?
3. Look up the conformal diagram of this solution and comment on it.

Conserved charges

After understanding the definition of mass, charge and angular momentum, show that the parameter M in the line element of the Schwarzschild black hole is indeed the mass.

Deflection of light

Consider a null geodesic incident from infinity on a Schwarzschild black hole. Let E and h denote the conserved quantities associated with the timelike Killing field and the angular Killing field $\partial/\partial\phi$.

1. Show that the maximum value for the impact parameter $b \equiv |h/E|$ for which the geodesic falls into the black hole is $b_{\max} = 3\sqrt{3}M$ (Hint: recall that photon circular orbits occur at $r = 3M$).
2. Determine the geometrical interpretation of the impact parameter (Hint: consider the $m \to 0$ limit).

Challenge Problem: Einstein-Rosen Bridge

Consider the Schwarzschild solution in (t, r, θ, ϕ) coordinates.

1. Define ρ via $r = \rho + M + \frac{M^2}{4\rho}$ (work with $G = 1$). For each value of r you should find two solutions: pick greater values of ρ for region I of the Kruskal diagram, and smaller values for region IV.

2. Calculate ds^2 in (t, ρ, θ, ϕ) coordinates.
3. Show that on surfaces of constant t the metric has the topology of $\mathbb{R} \times S^2$, where the proper radius of S^2 is r.
4. Considering $\theta = \frac{\pi}{2}$, draw a diagram of the resultant $\mathbb{R} \times S^1$ subsurface that connects regions I and IV.

0101 SCHWARZIELD GEOMETRY.

WE THEN HAVE :

1

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2 \ : \ w/\ f(r) = \left(1 - \frac{2M}{r}\right)$$

BASICALLY, WE HAVE TO STUDY EOM :

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -f(r)\dot{t}^2 + \frac{\dot{r}^2}{f(r)} + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

EOMS:

$$t)\quad d/d\tau\,(\overbrace{-2f(r)\dot{t}}^{E}) = 0\ !$$

$$r)\quad d/d\tau\left(\frac{2\dot{r}}{f(r)}\right) - \left(-f'(r)\dot{t}^2 + \frac{-f'(r)}{f(r)^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2\right)$$

$$\theta)\quad d/d\tau\,(2r^2\dot{\theta}) - 2r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\phi)\quad d_\tau\,(\underbrace{r^2\sin^2\theta\,\dot{\phi}}_{L}) = 0$$

OBSERVE $\exists$ TWO CONSERVED QUANTITIES ! SO :

$$\dot{t} = \frac{E}{f(r)}\ ; \qquad \dot{\phi} = \frac{L}{r^2\sin^2\theta}\ ;$$

(2) ONE CAN THINK THAT TO LOOK AT EOM FOR THIS CASE COULD BE THE WAY TO GO ... BUT IT IS EASIER IF WE LOOK AT $\mathcal{L}$! GIVEN PREVIOUS CONSERVED QUANTITIES.

$$\mathcal{L} = -1 = -\frac{f(r)E^2}{f(r)^2} + \frac{\dot{r}^2}{f(r)} + 0 + \frac{r^2\sin^2\theta \; L^2}{r^4 \sin^4\theta}$$

$$-1 = -\frac{E^2}{f(r)} + \frac{L^2}{r^2} + \frac{\dot{r}^2}{f(r)}$$

$$\left(\frac{E^2}{f(r)} - 1 - \frac{L^2}{r^2}\right) f(r) = \dot{r}^2 \qquad \text{SO:}$$

$$\dot{r} = \sqrt{\left(\underbrace{E^2 - 1}_{\text{Eff}} + \frac{2M}{r} - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right)}$$

(3) DOES THE PREVIOUS EXPRESSION NOT REMIND YOU OF SOMETHING LIKE $E = T + V \propto \cdot\cdot \dot{r}^2 + V(r)$?

IN FACT! $\left(\times \frac{m}{2}\right)$

$$\frac{1}{2} m\dot{r}^2 = \frac{1}{2} m \Big(\underbrace{E^2 - 1}_{\text{Eeff}} + \underbrace{\frac{2M}{r} - \frac{L^2}{r^2} f(r)}_{-V(r)} \Big) \Rightarrow$$

SO THE POTENTIAL:

$$\frac{1}{2}w\left[-\frac{2M}{r}+\frac{L^2}{r^2}\left(1-\frac{2M}{r}\right)\right]=V$$

WILL TELL US WHERE CIRCULAR ORBITS LIE. RECALL THAT IT MEANS THAT IT IS GIVEN BY THAT R WHERE $V'(R)=0$:

SO $V'(r)=0 \Rightarrow \frac{1}{2}w\left[\frac{2M}{r^2}-\frac{2L^2}{r^3}+\frac{2.3ML^2}{r^4}\right]=0$

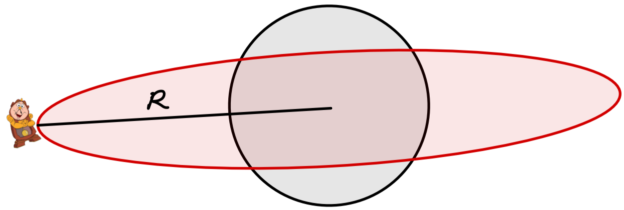
SO: $Mr^2-L^2r+3ML^2=0$; WE HAVE THEN TWO CIRCULAR ORBITS(

$$r_\pm=\frac{L^2\pm\sqrt{L^4-12M^2L^2}}{2M}=\frac{L^2\pm L\sqrt{L^2-12M^2}}{2M}$$

OBSERVE THAT $|L|\geq 2M\sqrt{3}$.

0102 CLOCK ORBITING BLACK HOLES

① WE HAVE EXACTLY THE SAME GEOMETRY AS PREVIOUS EXERCISE.



IF WE HAVE A FIXED RADIOUS $\Rightarrow \dot{r} = 0$. BUT HOW DO t AND τ RELATE TO EACH OTHER?

WE KNOW:

$$\dot{t} = \frac{dt}{d\tau} = \frac{E}{f(R)} \, ; \qquad \text{SO} \qquad t = \int d\tau \, \frac{E}{f(R)} \, ;$$

$\dot{t}$ = $\frac{E}{f(R)}$

CAN WE EXPRESS E IN TERMS OF GEOMETRY? YES! IF ORBIT $\equiv$ CIRCULAR $\rightarrow \dot{r} = 0$ SO

$$\dot{r} = \sqrt{\left(\underbrace{E^2 - 1}_{\text{Eff}} + \frac{2M}{r} - \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right)\right)}$$

$\dot{r} = 0$;

$\Rightarrow$

SO THE CIRCULAR ORBITS LIE AT:

$\dot{r} = 0 \Rightarrow$ SO $V'\big|_R = 0$:

$$0 = -\frac{2M}{R^2} + \frac{2L^2}{R^3}\left(1 - \frac{2M}{R}\right) - \frac{L^2}{R^2}\left(\frac{2M}{R^2}\right)$$

$$= -\frac{2M}{R^2} + \frac{2L^2}{R^3} - \frac{4ML^2}{R^4} - \frac{2ML^2}{R^4}$$

$$0 = -2MR^2 + R2L^2 - 6ML^2$$

$$0 = -MR^2 + RL^2 - 3ML^2$$

$$L^2 = \frac{MR^2}{(R-3M)}$$ $\Longrightarrow$ INTRODUCE IN $\dot{r}\big|_R = 0$ TO GET E:

$$E^2 = \frac{L^2}{R^2}\left(1 - \frac{2M}{R}\right) - \frac{2M}{R} + 1$$

$$= \frac{M\cancel{R^2}}{(R-3M)\cancel{R^2}}\left(1 - \frac{2M}{R}\right) - \frac{2M}{R} + 1$$

$$= \frac{M}{(R-3M)}\left(\frac{R-2M}{R}\right) - \frac{2M(R-3M)}{(R-3M)R} + \frac{(R-3M)R}{(R-3M)R}$$

$$= \frac{MR - 2M^2 - 2MR + 6M^2 + R^2 - 3MR}{(R-3M)R}$$

$$= \frac{-4MR + 4M^2 + R^2}{(R-3M)R} = \boxed{\frac{(R-2M)^2}{R(R-3M)} = E^2}$$

SO FINALLY,

$$dt = \frac{E}{-g_{tt}} d\tau \Rightarrow dt = \frac{(R-\cancel{2M})}{\sqrt{R(R-3M)}} \cdot \frac{R}{\cancel{R-2M}} d\tau$$

$$\boxed{\sqrt{(R-3M)/R} \quad dt = d\tau}$$ PROPER TIME OF THE CLOCK.

(1+2) THE TIME DILATION FOR AN OBSERVER AT "R" CAN BE VALID FOR AN OBSERVER SITTING AT $R = R_{clock}$ OR $R \to \infty$.

AS IT IS STATIC WE KNOW:

$$ds^2_{obs} = -d\tau_0^2 = -f(R_0)\, dt^2 ;$$

$$d\tau_0 = \sqrt{\frac{R_0 - 2M}{R_0}}\, dt$$

SO $$\boxed{d\tau_c = d\tau_0 \sqrt{\left(\frac{R-3M}{R_0-2M}\right)\frac{R_0}{R}}}$$

OBSERVE THAT $\Delta\tau_0 > \Delta\tau_c$ ALWAYS

- IF SITTING AT SOME ORBIT: $$d\tau_0 = \sqrt{\frac{R-2M}{R-3M}}\, d\tau_c$$

- IF OBS SITS AT $R_0 \to \infty$

$$d\tau_{0\infty} \cong \sqrt{\frac{R}{R-3M}}\, d\tau_c$$

TAKE TAYLOR

CHECK PREVIOUS PAGE

N81 REISSNER NORD BH.

WE ARE GIVEN THAT UGLY METRIC WITH A CHARGE AS:

$$ds^2 = -h(r)\,dt^2 + h(r)^{-1}\,dr^2 + r^2\,d\Omega_2^2$$

$$\text{w/} \quad h(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$$

BEFORE STARTING DOING THE EXERCISE, LET ANALISE WHERE THE HORIZON LIES.

$$h(r_H) = 0 \quad \longrightarrow \quad r_H^2 - 2GMr_H + GQ^2 = 0.$$

$$\text{SO} \qquad r_H = GM \pm \sqrt{(GM)^2 - Q^2G}$$

TWO HORIZONS. THE MINIMAL VALUE r_{H+} CAN TAKE HAPPENS WHEN $M^2 = \frac{Q^2}{G}$; THIS IS CALLED THE EXTREMAL CASE, AS:

$$r_H(\text{extr}) = GM \implies \text{SO THE EXTREME METRIC.}$$

$$h(r)\big|_{ext} = 1 - \frac{2r_{ext}}{r} + \frac{r_{ext}^2}{r^2} = \left(1 - \frac{r_{ext}}{r}\right)^2 =$$

$$= \left(1 - \frac{GM}{r}\right)^2$$

(WHICH THE SAME METRIC AS NEXT EXERCISE).

I WILL REFER TO $h(r)\big|_{ext}$ AS $h(r)$.

① THE CONSERVED QUANTITIES ARE:

$$L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -h(r)\dot{t}^2 + h(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + \dots$$

$t)$: $\frac{d}{d\tau}\left(-h(r)\dot{t}\right) = \frac{d}{d\tau}E = 0$ (CONSERVED)

$r)$ $\frac{d}{d\tau}\left(2h(r)^{-1}\dot{r}\right) - \partial_r L = 0$

$\theta)$ $\frac{d}{d\tau}\left(2r^2\dot{\theta}\right) - r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$

$\phi)$ $\frac{d}{d\tau}\left(r^2\sin^2\theta\,\dot{\phi}\right) = \frac{d}{d\tau}(L) = 0$ (CONSERVED)

② RADII? CLASSIC $E = T + V$ PROBLEM.

$$-1 = -h(r)\dot{t}^2 + h(r)^{-1}\dot{r}^2 + \underbrace{\cancel{r^2\dot{\theta}^2}}_{\theta=\pi/2} + r^2\underbrace{\sin^2\theta}_{1}\dot{\phi}^2$$

$$-1 = -\frac{E}{h(r)} + \frac{\dot{r}^2}{h(r)} + \frac{L^2}{r^2}$$

$$E = \dot{r}^2 + \underbrace{h(r)\left(1 + \frac{L^2}{r^2}\right)}_{V_{pot}} \Leftarrow \text{CLASSIC}$$

$$V_{pot} = h(r)\left(1+\frac{L^2}{r^2}\right);$$

CIRCULAR ORBITS $\longrightarrow$ $\partial_r V = 0$

$$0 = h'(r)\left(1+\frac{L^2}{r^2}\right) - \frac{2h(r)L^2}{r^3}$$

$$0 = 2\left(1-\frac{GM}{r}\right)\left(+\frac{GM}{r^2}\right)\left(1+\frac{L^2}{r^2}\right) - \frac{2L^2}{r^3}\left(1-\frac{GM}{r}\right)^2$$

$$= \frac{2GM}{r^2}\left(1+\frac{L^2}{r^2}-\frac{GM}{r}-\frac{GML^2}{r^3}\right) - \frac{2L^2}{r^3}\left(1+\frac{G^2M^2}{r^2}-\frac{2GM}{r}\right)$$

$$= \frac{2GM}{r^2} + \frac{2GML^2}{r^4} - \frac{2(GM)^2}{r^3} - \frac{2(GM)^2L^2}{r^5}$$

$$+ \frac{4GML^2}{r^4} - \frac{2L^2}{r^3} - \frac{2(GM)^2L^2}{r^5}$$

$$= 2GMr^3 - 2r^2\left((GM)^2+L^2\right) + 6GML^2 r - 4(GM)^2L^2 = 0$$

$\Rightarrow$ SOLVE :

$r = GM$ $\Longleftarrow$ BUT THIS THE HORIZON.

$$r_\pm = \frac{L^2 \pm L\sqrt{L^2 - 8(GM)^2}}{2GM};$$

OUTER IS ALWAYS MORE STABLE.

r_- IS INSIDE HORIZON.

③ FOR A γ, WE JUST NEED TO REPEAT PREVIOUS COMPUTATION WITH $L=0$.

$$0 = -h(r)\dot{t}^2 + h(r)^{-1}\dot{r}^2 + r^2 \cancel{\dot{\theta}^2}_{\theta=\pi/2} + r^2 \underbrace{\sin^2\theta}_{1} \dot{\phi}^2$$

$$0 = -\frac{E}{h(r)} + \frac{\dot{r}^2}{h(r)} + \frac{L^2}{r^2}$$

$$E = \dot{r}^2 + \underbrace{h(r)\left(\frac{L^2}{r^2}\right)}_{V_\gamma}$$

SAME STORY: $\partial_r V = 0$

$$h'\left(\frac{L^2}{r^2}\right) + h(r)\left(-\frac{2L^2}{r^3}\right) = 0$$

$$4(GM)^2 L^2 - 6\, GM\, L^2 r + 2L^2 r^2 = 0 \, ;$$

$$r_- = GM \text{ (HORIZON)} \quad ; \quad r_+ = 2GM$$

$$r_\pm \text{ (NO EXTREMAL)} = \frac{3r_S}{4} \pm \sqrt{\left(\frac{3r_S}{4}\right)^2 - 2Q^2}$$

(4) EXTRA WHAT IF WE CONSIDER A CHARGED MASSIVE PARTICLE?

IF THAT WAS THE CASE, WE WOULD REQUIRE TO MODIFY THE Lagrangian. AS WE HAVE A PARTICLE COUPLED TO $A^\mu = (\phi, \bar{A})$, THIS EM SHOULD APPEAR IN LAGRANGIAN.

$$\mathcal{L} = u_\mu u^\mu - q A_\alpha u^\alpha$$

$$= g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - q A_\alpha \dot{x}^\alpha$$

WE DO NOT WANT TO MAKE IT HARD, SO ASSUME A COMPLETE RADIAL DISTRIBUTION OF $\bar{E}$ AND $\bar{B} = 0$ (i.e. $\bar{A} = 0$)

SO $A = \phi \propto \frac{Q}{r}$

SO THE LAG IS

$$\mathcal{L} = -f\dot{t}^2 + f^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 - \frac{qQ}{r}\dot{t}$$

$\Rightarrow$

IMPLIES $\Rightarrow$ CONSERVED QUANTITIES FOR EQUATORIAL PLANE ORBI.

$$\partial_{\dot t} \mathcal{L} = f^2 \dot t + \frac{qQ}{r} = \hat E \Rightarrow f^2 \dot t = \left(\hat E - \frac{qQ}{r}\right)$$

$$\partial_{\dot\phi} \mathcal{L} = r^2 \dot\phi = L$$

REINTRODUCE IN LAG:

EQUATOR

$$\mathcal{L} = -f^2 \dot t^2 + f^{-2} \dot r^2 + \cancel{r^2 \dot\theta^2} + r^2 \sin^2\theta\, \dot\phi^2 - \frac{qQ}{r} \dot t$$

$$= -\dot t \left(f^2 \dot t - \frac{qQ}{r}\right) + f^{-2} \dot r^2 + \frac{L^2}{r^2}$$

$$= -\left(\frac{\hat E - qQ}{r}\right) \frac{1}{f^2} \hat E + \frac{1}{f^2} \dot r^2 + \frac{L^2}{r^2} = -1$$

$$= -\hat E^2 + \frac{qQ}{r} \hat E + \dot r^2 + \left(\frac{L^2}{r^2} + 1\right) f^2(r) = 0$$

$$\Rightarrow \hat E^2 = \dot r^2 + \underbrace{\frac{qQ}{r} \hat E + \left(\frac{L^2}{r^2} + 1\right) f^2(r)}_{-V} = 0$$

ORBITS $\Rightarrow$ $\partial_r V = 0$ + $\dot r = 0$ $\Rightarrow$ LEFT AS AN EXERCISE.

EXTREMAL CHARGED BLACK HOLE.

FROM PREVIOUS EXERCISE, WE KNOW THAT:

$$ds^2 = -f(r)^2 dt^2 + f(r)^{-2} dr^2 + r^2 d\Omega_2^2$$

① THE PREVIOUS METRIC ONLY COVERS THE PATCH OF SPACE SUCH THAT $r > M$. BUT IF WE DO A "EF" COORDINATE TRANS WE CAN COVER THAT REGION SOMEHOW.

"EF" MEANS THAT $dr_* = \dfrac{dr}{f(r)^2}$.

WE THEN DEFINE $v = t + r^*$ (INGOING)

$$ds_{EF}^2 = -f(r)^2 dv^2 + 2dvdr + r^2 d\Omega_2^2$$

INTEGRATING $dr_* \Rightarrow r + 2M\log(r-M) - \dfrac{M^2}{r-M} = r_*$

OBSERVE THAT THERE IS NO SINGULARITY AT $r=M$.

(2) TO COMPUTE THE PROPER LENGTH, RECALL THAT ds IS THE ONE IN CHARGE FOR IT. AS (t, θ, ϕ) ARE FIXED, WE HAVE:

RN

$$ds^2 = \frac{dr^2}{f(r)^2} \Rightarrow S = \int_M^{r_0} \frac{dr}{1 - M/r} = \int_M^{r_0} \frac{r}{r-M}\, dr$$

$\Rightarrow$ CHANGE VARIABLES $r' = r - M$ SO:

$$\int_0^{r_0'} \frac{r'+M}{r'}\, dr' \Rightarrow r' + M \log r' \Big|_0^{r_0'} = \infty$$

BUT, IT IS NOT PRESENT IN SCHWARZSCHILD.

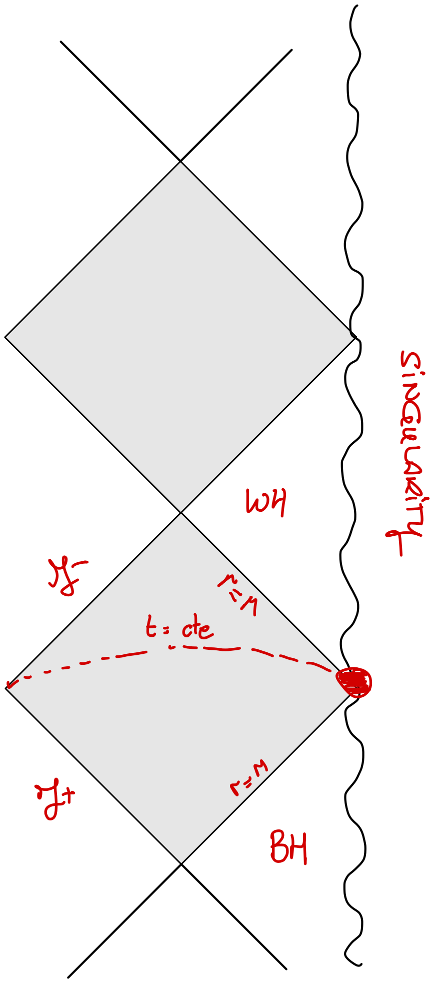
SCH

$$\int ds = \int_{2M}^{r_0} \frac{dr}{\sqrt{1 - \frac{2M}{r}}} =$$

$$= \sqrt{1 - \frac{2M}{r_0}}\, r_0 + 2M \,\mathrm{ArcTanh} \sqrt{1 - \frac{2M}{r_0}} - 0$$

$\Rightarrow$ SOMETHING FINITE.

ⓒ THE CONFORMAL DIAGRAM.



- INGOING LIGHT HITS THE SINGULARITY
- TIMELIKE OBSERVERS CAN AVOID THE SINGULARITY.
- $T = cnt$ IS THE INFINITE THROAT ALL THE WAY TO $r=M$.

0.11 CONSERVED CHARGES

IN ORDER TO GIVE AN APPROPIATED SOLUTION FOR THIS PROBLEM, WE NEED SEVERAL CONCEPTUAL BULLET POINTS:

- CONSERVED QUANTITIES ARE LIKE CHARGES; THEY ARE UNDERSTOOD BY LOOKING AT FLUXES OUTWARD "HYPERSURFACES".

- STOKES THEOREM QUITE USEFUL; RECALL:

$$\int_M dx^n x \sqrt{|g|}\, \nabla_\mu V^\mu = \int_{\partial M} dy^{n-1} \sqrt{|h|}\, n_\mu V^\mu$$

WITH:
- x = COORDINATES IN BULK (n-dim SPACE)
- y = " IN HYPERSURFACE ($(n-1)$dim SPACE)
- g = BULK METRIC
- h = INDUCED METRIC (RESTRICTED g IN TERMS OF y)

- CONTRACTING W/ KILLING V GIVES CONSERVED QUANTITIES.

FOR EXAMPLE, THE ELECTRIC CHARGE Q:

$$Q = -\int_{M} d^3x \sqrt{h} \; n_\mu J^\mu \qquad w/ \; n_\mu = (1,0,0,0)$$

BUT $J^\mu = \nabla_\mu F^{\mu\nu}$ (FROM EM).

APPLY STOKES TO $Q \Longrightarrow \int_{\partial M} - \sqrt{h^{(2)}} \; n_\mu \sigma_\nu F^{\mu\nu}$

w/ $h^{(2)}$ INDUCED METRIC ON A 2-SPHERE. AND σ_ν THE 3D normal $\vec{\sigma}$ FOR SPHERE.

THE ENERGY:

ONE MAY THINK THAT SOMETHING LIKE A CONTRACTION OF ENERGY MOMENTUM TENSOR WITH A KILLING CAN GIVE US A CONSERVED QUANTITY ... $(T^{\mu\nu} K_\nu)$ BUT NOT ALWAYS TRUE. IN SCHWARZSCHILD IS 0 ... WE NEED SOMETHING MORE INVOLVED... FOR EXAMPLE:

$$J_R^\mu = 8\pi G \, K_\nu \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) = K_\nu R^{\mu\nu}.$$

OBSERVE THAT IS THE Ricci WHAT WE WROTE!

MANIPULATING WE HAVE:

$$J^\mu_R = K_\nu R^{\mu\nu} = \nabla_\mu \nabla_\nu K^\mu ;$$

THE DEFINITION OF ENERGY IS:

STOKES

$$E = \frac{1}{4\pi G}\int_M d^3x \sqrt{h}\; n_\mu \nabla_\nu \nabla^\mu K^\nu =$$

$$\frac{1}{4\pi G}\int_{\partial M} d^2y \sqrt{h^{(2)}}\; n_\mu \sigma_\nu \nabla^\mu K^\nu .$$

WHAT IS E FOR SCHWARZIED GEOMETRY?

THE NORMAL VECTOR OF THE 4D SPACE IS TIME.

$$n_\mu = \left[\left(1-\frac{2GM}{r}\right)^{1/2}, \vec{0}\,\right]$$

THE NORMAL VECTOR OF THE SURPHACE IS $\sigma = \left(0, \left(1-\frac{2GM}{r}\right)^{-1/2}, 0, 0\right)$

SO:

KILLING OF TIME.

$$n_\mu \sigma_\nu \nabla^\mu K^\nu = -\nabla^0 K^1 = -\nabla^0 (1,0,0,0)$$

$$-\nabla^0 k^1 = -g^{00}\nabla_0 k^1 = -g^{00}\left(\cancel{\partial_0 k^1}^{\,0} + \Gamma^1_{0\lambda} k^\lambda\right) =$$

$$= + f(r)^{-1} \cdot \Gamma^r_{tt} \, f(r) = + \frac{GM}{r^2} .$$

FOR THE 2D INDUCED METRIC WE HAVE: $\sqrt{h^{(2)}} = r^2 \sin\theta$

SO, AT THE END:

$$E = \frac{1}{4\pi G}\int_{\partial M} d\theta\, d\phi \; r^2 \sin\theta \cdot \frac{GM}{r^2} = \frac{4\pi G}{4\pi G} M ;$$

RESTORE "C" TO SEE : $\boxed{E = Mc^2}$ INDEED M IS THE MASS.

FOR THE ANGULAR MOMENTA:

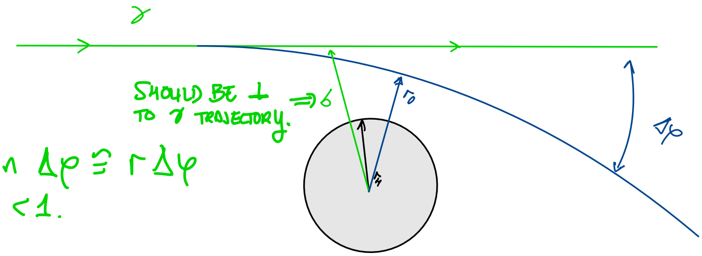
$$L = -\frac{1}{8\pi G}\int_{\partial M} d^2x \sqrt{h^{(2)}} \; n_\mu \sigma_\nu \nabla^\mu K^{(2)\nu}$$

WITH $\chi^\mu_\phi = K^{(2)}{}_\nu R^{\mu\nu}$.

N82 DEFLECTION OF LIGHT

1

WE WANT TO COMPUTE THE MAXIMAL VALUE THE IMPACT PARAMETER "b" CAN TAKE. LET'S PICTURE WHAT WE HAVE:



$b \equiv r \sin \Delta\varphi \cong r\Delta\varphi$
if $\Delta\varphi < 1$.

AND WE KNOW THAT $b = h/E$. BASICALLY THIS IS AN OPTIMIZATION PROBLEM. AT WHICH POINT r CAN WE STILL HAVE $\dot{r} = 0$? (SO NO FALLING INTO BH?)

RECALL EOM IN SCHWARZSCHILD AFTER USING CONSERVED QUANTITIES.

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1} dr^2 + r^2 d\Omega_2^2 \dots$$

$$\dot{r}^2 = E^2 - f(r)\frac{h^2}{r^2}; \quad \text{BUT } b = h/E.$$

$$\dot{r}^2 = E^2\left(1 - \frac{f(r)\,b^2}{r^2}\right); \quad \text{SOLVE FOR } \dot{r} = 0 \Longrightarrow$$

$b^2 = \dfrac{r^2}{f(r)}$; b_{max} WILL HAPPEN AT THE MINIMAL ORBIT A

γ CAN HAVE FOR SCHWARZSCHILD, i.e. $r_0 = 3M$.

SO $b_{max} = \sqrt{\dfrac{9M^2}{\frac{3M-2M}{3M}}} = \sqrt{27M^2} = 3\sqrt{3}\,M$ ✓

(2) OBSERVE THAT $M \to 0$ MEANS $b^2 = \dfrac{r_{orbit}^2}{1 - \cancel{\frac{2M}{r}}} = r_{orbit}^2$;

SO $b(M=0) = r_0$ WHICH CORRESPONDS TO THE CLOSEST

APPROACH OF THE GEODESIC IN THE ABSENCE OF GRAVITY

Gravitational Waves

Gravitational waves from binary

Consider two black holes of equal mass M rotating around each other on a circular orbit of radius R. This system generates gravitational waves, given by

$$\bar{h}_{ij} = \frac{2G}{r}\frac{d^2 I_{ij}(t-r)}{dt^2}, \tag{1}$$

where I_{ij} is the quadrupole momentum tensor

$$I^{ij}(t) = \int d^3y\, y^i y^j T^{00}(t,\vec{y}). \tag{2}$$

y^i are spatial Cartesian coordinates on flat space, $h_{\mu\nu}$ is a perturbation about flat space, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ and we are going to work in the gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$.

1. Use Newtonian mechanics to find the angular velocity Ω of the stars as a function of M, R.
2. Compute I_{ij} for the above binary system, where the stress-energy tensor $T^{00}(t,\vec{y})$ is simply given by a product of delta functions at the instantaneous location of the stars.
3. Compute all components of $\bar{h}_{ij}$ for this system in the given gauge.
4. By going to transverse traceless gauge along the z-axis, compute the metric perturbation h^{TT}_{ij} and find frequency, amplitude and polarisation of gravitational waves.
5. Find the total power radiated by gravitational waves (Recall: $P = -\frac{G}{5}\left\langle \partial_t^3 Q_{ij} \partial_t^3 Q^{ij}\right\rangle$ where $Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk}$).

Time to Merger

Starting from the previous exercise, we will now assume that $R = R(t)$. Since the system is emitting gravitational radiation, it is losing energy and as a consequence the orbit is shrinking. At some point, the black holes will collide and merge. In this exercise, we will compute the time it takes for that to happen, starting at an initial separation $R(t=0) = R_0$. We work in the Newtonian approximation.

1. Compute the total energy E of the binary system, by adding kinetic and gravitational potential energy.
2. By imposing $\frac{dE}{dt} = P$, where P is as found in exercise 1 , solve for $R(t)$.
3. Using the solution above and assuming $R(T) = 0$, show that:

$$T = \frac{5R_0^4}{32(GM)^3}. \tag{3}$$

4. Suppose $M \simeq 30 M_\odot$. What is T if $R_0 = 1$AU? And what is R_0 if $T = 1$ year?

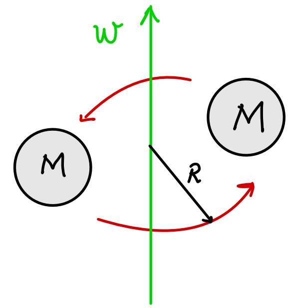
Challenge Problem: Gravitational Wave Detection

Consider a freely falling observer with four-velocity u^μ. In flat space, $u^\mu = (1,0,0,0)$ and the (x, y, z) axes do not vary with time. The curved spacetime equivalent of this is the so-called parallely-transported frame. We can define orthonormal axes e_i^μ that obey $\nabla_u e_i = 0$. Also note that u is geodesic, i.e. $\nabla_u u = 0$. The "physical distances" measured by the observer will be measured with respect to such a frame.

1. By contracting e_i^c with the geodesic deviation equation, find an expression for the relative acceleration between geodesics as measured by a freely falling observer.
2. At leading order we can consider $u^\mu = (1,0,0,0), e_1^\mu = (0,1,0,0)$ etc. In this case (which is only an approximation!) there is no difference between the indices i and μ. Show that the above equation reduces to
$$\frac{d^2 S_\mu}{d\tau^2} \simeq R_{\mu 00\nu} S^\nu. \tag{4}$$
3. Consider the following plane wave solution:
$$h_{\mu\nu} = \mathrm{Re}\left(H_{\mu\nu} e^{i\omega(z-t)}\right),$$
$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & 0 & 0 \\ 0 & 0 & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5}$$
Find explicit solutions for $S_1(\tau)$ and $S_2(\tau)$. Give an interpretation of your results. (Hint: the linearised Riemann tensor is given by $R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\nu\sigma,\mu\rho} - h_{\mu\rho,\nu\sigma}\right)$)

GRAVITATIONAL WAVES FROM BINARY SISTEM

WE HAVE THE FOLLOWING SYSTEM THAT GENERATES A PERTURBATION IN THE FORCE AS:



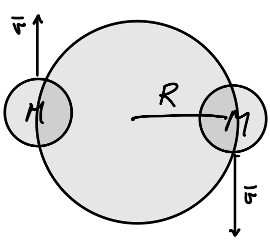
$$\bar{h}_{ij} = \frac{2G}{r} \frac{d^2 I_{ij}(t-r)}{dt^2},$$

WITH $I_{ij}(t) = \int d^3y \; y^i y^j \, T^{00}(t, \bar{y})$.

$$h^{TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & h_{ij} & \\ 0 & & & \end{bmatrix}$$

- $y^i \equiv$ COORDINATES IN FLAT SPACE;
- $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \, \eta_{\mu\nu}$

① WHAT IS THE ANGULAR VELOCITY? (USE NEWTONIAN MECH)



$$\bar{v} = v \cdot \begin{pmatrix} \cos \Omega t \\ \sin \Omega t \end{pmatrix} \Rightarrow \bar{a} = \Omega v \begin{pmatrix} -\sin \Omega t \\ \cos \Omega t \end{pmatrix}$$

RECALL THAT $a = \dfrac{v^2}{R}$

SO:

$$a = |\bar{a}| = \sqrt{(\Omega v)^2 (s^2 + c^2)} = \Omega v.$$

WE ALSO KNOW THAT GRAV ATTRACTION IS BALANCED WITH CENTRIFUGAL FORCE :

$$F_G = F_C \Rightarrow \frac{G M^{\cancel{2}}}{(2R)^2} = \cancel{M} . a = \frac{v^2}{R} \Rightarrow$$

DISTANCE BTW BH

$$\Rightarrow v = \sqrt{\frac{GM}{4R}} \Rightarrow \text{BUT } \Omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi R}{v}} \Rightarrow \left[\Omega^2 = \frac{GM}{4R^3} \right]$$

(2) COMPLETE I_{ij} W/ $T^{00} \equiv$ LC OF STARS LOCATION.

OBSERVE THAT THE POSITION OF STARS IS OPPOSITE SO:

$\bar{x}_s(t) = R \begin{bmatrix} \cos \Omega t \\ \sin \Omega t \end{bmatrix}$; WE SHOULD USE $\delta^3(\bar{x} - \bar{x}_s)$ TO DESCRIBE WHERE THE MASS IS LOCATED AT EACH TIME.

$$\text{SO} \quad T^{00}(t, \bar{x}) = M\left(\delta^3(\bar{x} - \bar{x}_{s_1}) + \delta^3(\bar{x} + \bar{x}_{s_1}) \right)$$

ASSUME THE STARS ARE LOCATED IN THE XY PLANE SO:

$$I^{ij}(t) = \int d^3\bar{y} \; y^i y^j \, M\left(\delta^3(\bar{x} - \bar{x}_{s_1}) + \delta^3(\bar{x} + \bar{x}_{s_2}) \right)$$

$$= 2MR^2 \begin{bmatrix} c^2 \Omega t & c\Omega t \, s\Omega t & 0 \\ c\Omega t \, s\Omega t & s^2 \Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

WHERE $c\theta s\theta = \frac{1}{2} \sin 2\theta$

(3) LET'S COMPUTE DERIVATIVES OF THE QUADRUPOLE MOMENTUM.

ASSUME THAT $t \longrightarrow (t-r)$ FOR THE ARGUMENT.

USE DOUBLE ANGLE TRIGO.

$$\dot{I} = 2MR^2\Omega \begin{bmatrix} -s\,2\Omega t & c\,2\Omega t & 0 \\ c\,2\Omega t & s\,2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ddot{I} = 2MR^2\Omega\,(2\Omega) \begin{bmatrix} -c\,2\Omega t & -s\,2\Omega t & 0 \\ -s\,2\Omega t & c\,2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dddot{I} = 2MR^2\Omega\,(2\Omega)^2 \begin{bmatrix} s\,2\Omega t & -c\,2\Omega t & 0 \\ -c\,2\Omega t & -s\,2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

FOR LAST SECTION

$(t-r)$

SO $$\bar{h}_{ij} = \frac{8GMR^2\Omega^2}{r} \begin{bmatrix} -c\,2\Omega t_* & -s\,2\Omega t_* & 0 \\ -s\,2\Omega t_* & c\,2\Omega t_* & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) WE WANT TO COMPUTE $\bar{h}_{ij}^{TT}$ IN THE TRANSVERSE - TRACELESS GAUGE IN THE z-DIRECTION. THIS MEANS THAT WE WANT TO SEE THE "OSCILLATIONS" PROJECTED ONTO THE z-PLANE:

FOR THAT WE NEED TO KNOW HOW THE PROJECTION OF AN OBJECT IN $\mathbb{R}^3$ LOOKS LIKE:

- ASSUME $X_{ij} \equiv$ TENSOR IN $\mathbb{R}^3$.

$$X_{ij}^{TT} = \left(P_i^{\ k} P_j^{\ l} - \frac{1}{2} P_{ij} P^{kl} \right) X_{kl}$$

WHERE $P_{ik} = \delta_{ik} - n_i n_k$,

WITH $n_i \equiv$ NORMAL VECTOR IN THE DIRECTION OF OBSERVATION.

TO SIMPLIFY COMPUTATIONS, LET'S MOVE TO MATRIX NOTATION:

OBSERVE THAT CONTRACTIONS OF INDICES STATE THAT:

transpose

$$\underbrace{X^{TT}}_{\text{transverse - traceless.}} = P^T X P - \frac{1}{2} P \, \mathrm{Tr}(PX)$$

$$\text{BUT } P^T = P = \mathbb{1} - (0,0,1) \overbrace{(0,0,1)}^{n_i} =$$

$$= \mathbb{1} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

AND $X^{TT} \equiv$ OUR $\bar{h}_{ij}$

OBSERVE THAT $P \bar{h} P = \bar{h}$ AND $P \bar{h} = \bar{h}$

SO $\mathrm{Tr}(P\bar{h}) = \mathrm{Tr}(\bar{h}) = 0$ $\Leftarrow$ BECAUSE $\bar{h}$ IS NOT IN LORENTZ SIGN.

$\Rightarrow$ THIS ALL MEANS THAT $h_{ij}^{TT}(t,r) = \bar{h}_{ij}(t,r)$

$$\bar{h}_{ij} = \underbrace{\frac{8 G M R^2 \Omega^2}{r}}_{\propto} \begin{bmatrix} -c\,2\Omega t_* & -s\,2\Omega t_* & 0 \\ -s\,2\Omega t_* & c\,2\Omega t_* & 0 \\ 0 & 0 & 0 \end{bmatrix} = h_{ij}^{TT}$$

$(t-r)$

AS WE NOW HAVE THE (TT) PERTURBATION, WE CAN READ OF ITS FEATURES:

FREQUENCY: ARGUMENT OF TRIG $\Rightarrow \omega = 2\Omega$.

AMPLITUDE: WHAT IS INFRONT OF MATRIX $\Rightarrow \propto$

POLARISATION: IT IS A MIXED ONE, AS THERE ARE OUT OF DIAG TERMS; RECALL THAT

$$h_{ij}^{TT} = (A)\text{ POLARISATION } + \;+\; (B)\text{ POLARISATION } \times$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix} h_+ & & 0 \\ 0 & h_+ & \\ & & \ddots \end{pmatrix} \qquad \begin{pmatrix} 0 & h_\times & h_\times' \\ h_\times & 0 & \\ h_\times' & & \ddots \end{pmatrix}$$

IN OUR CASE $h_+ \sim \cos \omega t$, $h_\times \sim \sin \omega t$.

$\boxed{5}$ RECALL THAT WE COMPUTED $\dddot{I}$ FOR SOMETHING:

$$\dddot{I} = \underbrace{2MR^2\,\Omega\,(2\Omega)^2}_{\beta} \begin{bmatrix} s\,2\Omega t & -c\,2\Omega t & 0 \\ -c\,2\Omega t & -s\,2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

AND THE RADIATED POWER IS:

$$P = -\frac{G}{5}\left\langle \partial^3_t Q_{ij}\, \partial^3_t Q^{ij} \right\rangle \quad \text{w/} \quad Q_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I_{kk}$$

WITH $I_{kk} \equiv Tr(I_{ij})$

IN OUR CASE $Tr(I_{ij}) = 0$!

SO $Q_{ij} = I_{ij} \Rightarrow \partial^3_t I_{ij} = \dddot{I}_{ij}$

NOTICE THAT, AS WE ARE DEALING W/ SPATIAL METRICS

WE THEN HAVE: $I_{ij} = I^{ij}$ SO

$$\dddot{I}_{ij}\dddot{I}_{ij} = \beta^2\left(s^2\omega t + c^2\omega t + c^2\omega t + s^2\omega t\right)$$

$$= 2\beta^2 \neq f(t)$$

SO $\langle \dddot{I}_{ij}\dddot{I}^{ij} \rangle$ = NO AVERAGE OVER t $\Rightarrow$ $\dddot{I}_{ij}\dddot{I}^{ij}$

(BECAUSE NO TIME DEP)

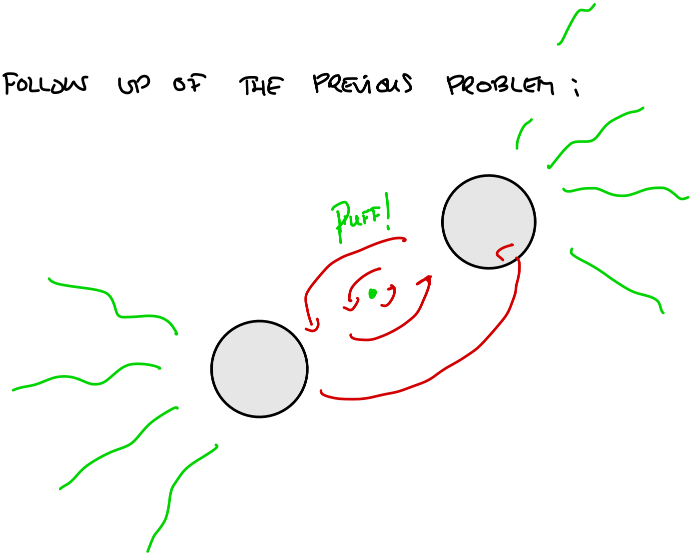
SO WE HAVE $P = -\frac{G}{5} \beta^2 = -\frac{G}{5} 128 M^2 R^4 \Omega^6$

BUT RECALL $\Omega^2 = \frac{GM}{4R^3} \Rightarrow$

$$\boxed{P = -\frac{2}{5} \frac{G^4 M^5}{R^5}}$$

TIME TO MERGE

THIS IS A FOLLOW UP OF THE PREVIOUS PROBLEM:



① IN NEWTON APPROX, COMPUTE E:

$$E = E_c + E_G :$$

$$E_G = -\frac{GM^2}{R} \quad ; \quad E_c = \frac{1}{2} M v^2$$; RECALL FROM PREVIOUS EXERCISE THAT:

$$v^2 = \frac{GM}{R}$$ (CAREFULL RESPECT TO CENTER MASS)

$$\text{SO} \quad E_T = \underbrace{\frac{GM^2}{2R}}_{E_c} - \underbrace{\frac{GM^2}{R}}_{E_G} = -\frac{GM^2}{R} \; !$$

(2,3) WE KNOW THAT $P = \partial_t E$ w/ P :

$$P = -\frac{2}{5}\frac{G^4 M^5}{R^5} = \frac{dE}{dt}$$

SO OUR $E = \frac{-GM^2}{2R} \Rightarrow$ APPLY $\frac{d}{dt}$:

$$\partial_t E = \frac{GM^2}{4R^2}\frac{dR}{dt} \; ;$$ COMPARE TO P :

$$\Rightarrow \frac{dR}{dt} = -8/5 \left(\frac{GM}{R}\right)^3$$

AND INTEGRATE: CONSIDER CONDITIONS: $t=0$, $R(t=0)=R_0$

$t=T$, $R(t=T)=0$

$$\int_{R_0}^{0} dR\, R^3 = -\int_0^T dt\, \frac{8}{5}(GM)^3 dt \Rightarrow$$

$$\Rightarrow \frac{R_0^4}{4} = \frac{8}{5}(GM)^3 T \Rightarrow \boxed{T = \frac{5R_0^4}{32(GM)^3}}$$

(4) USE A CALCULATOR TO SHOW :

- $R_0 = 1\,AU$, $M = 30\,M_\odot$ $\Rightarrow T \sim 10^{14}$ years
- $R_0 = ?$ IF $T \sim 1$ year $\Rightarrow R_0 = 5 \cdot 10^4$ KM.

Cosmology

To Quasars and beyond

Quasars are extremely luminous galaxies with energy fed by a supermassive black hole at their center. Since they are so luminous, they can be seen over very large redshifts. Assume that a quasar has been seen at $Z = 6$. For this problem, assume that we are in a matter dominated universe at critical density. Assume that the Hubble constant today is $H_0 = 70$ km/s/Mpsc.

1. Find the age of the universe when the quasar emitted the light that we observe today.
2. Find the proper distance to this quasar.

Redshift

It is sometimes convenient to give the past Hubble constant as a function of $Z, H(Z)$. This means that $H(Z)$ is the Hubble constant at the time when the light was emitted that today have been redshifted with a factor Z. Suppose todays Hubble constant is H_0.

1. Find $H(Z)$ for a matter dominated universe.
2. Find $H(Z)$ for a vacuum dominated universe, for flat universe.
3. For a general $H(Z)$ find the deceleration parameter q in terms of $Z, H(Z)$ and $H'(Z)$.

Flat expanding universe

Consider a flat, $k = 0$, expanding universe, with constant deceleration parameter q.

1. Find a relation between q and the equation of state parameter w.
2. Find the Hubble constant as a function of the scale factor a, q, the Hubble constant today, H_0, and the scale factor today a_0.
3. Find the proper distance $D = a_0 r$ to a source in terms of the redshift factor Z, H_0 and q.

Matter domination

Assuming that today $\rho_m = \rho_{\text{crit}}$ and $\rho_r = 5 \times 10^{-14}$ J/m^3, find the age of the universe when it crossed over from radiation domination to matter domination. Today's Hubble constant can be assumed to be $H_0 = 70$ km/s/Mpsc.

Quintessence Element

Assume our present day universe has a Hubble constant H_0 and a matter components with energy density given by the critical density. Let us further assume that there is another substance in the universe called "quintessence" with energy density $\rho_q > 0$ and equation of state parameter $w = -1/3$.

1. Determine whether the universe is open or closed.
2. Determine the evolution of the universe. Will it keep on expanding?

Big Bounce, Big Rip, No Big Bang and Maybe Big Mac

Determine the motion[1] and fate of the following universes that contain matter and cosmological constant.

1. $(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$,
2. $(\Omega_M, \Omega_\Lambda) = (3, 0.1)$,
3. $(\Omega_M, \Omega_\Lambda) = (0.3, 2)$.

Boring Universe

Compute the luminosity distance, angular distance and the age (as a function of H_0) in an empty Universe, with $\Omega_i = 0$.

Universal Merry Go Round

1. Determine the maximum value of the comoving distance that a photon can travel from the Big Bang to the collapse moment in an Universe with matter domination and positive curvature k. How many periods is the photon able to perform before the Universe collapses?
2. Compute exactly the same, but consider now an Universe with radiation domination.

Chapters of Universe chronology

1. Consider a flat Universe with matter and cosmological constant Λ. Determine the redshift value z when the accelerated expansion started.
2. Specify for the case $\Omega_M = \Omega_\Lambda$ and compare both cases.

[1] classical.

QUASARS AND BEYOND.

$z=6$; ASSUME Ω_M DOMINATION AT CRITICAL.

① $t_{emission}$?

RECALL THAT FROM FRIEDMANN AND ρ_M DOMINATING + $\rho_M = \rho_{crit}$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{tot} - \frac{K}{a^2}$$

$\rho_{tot} = \rho_{matter} + \rho_{rad}$; BUT AS $\rho_m = \rho_{crit} \Rightarrow$

$\Omega_m \simeq 1 \rightarrow \Omega_r \simeq 0$ AND $K \simeq 0$.

$$\frac{\dot a}{a} = \sqrt{\frac{8\pi G}{3}\rho_{m_0}\left(\frac{a_0}{a}\right)^3}$$

$$\dot a = \sqrt{\frac{8\pi G}{3}\cdot\frac{3H_0^2}{8\pi G}\,\frac{1}{a^3}\cdot a^2}$$

$$\dot a = H_0\frac{1}{\sqrt a} \Rightarrow \int_{a_0}^{a}\sqrt a\,da = \int_{t_0} H_0\,dt \Rightarrow$$

t/t_0

$$\Rightarrow \frac{2}{3}a^{3/2} \propto H_0 t \Rightarrow a(t) = a_0\left(\frac{3}{2}H_0 t\right)^{2/3}$$

so $$1+z = \frac{a_0}{a_{emis}} \simeq$$

$$z = \left(\frac{2}{3H_0 t}\right)^{2/3} - 1 \Rightarrow t = (1+z)^{-3/2} \frac{2}{3H_0}$$

$$\boxed{t \simeq 5 \cdot 10^8 \text{ YEARS OLD WHEN EMITTED.}}$$

(2) YOU CAN FIND A FORMULA TO COMPUTE PROPER DISTANCES IN "FLAT EXPANDING" UNIVERSE. THIS IS:

$$D = \frac{-1}{qH_0}\left(\frac{1}{(1+z)^q} - 1\right)$$

"q" IS THE DECELERATION PARAMETER AS (SAME PROBLEM)!

$$q = 1/2\,(1+3w) \quad ; \text{ FOR MATTER } w=0$$

so $$D = \frac{2}{H_0}\left(1 - \frac{1}{(1+z)^{1/2}}\right) \simeq 18 \text{ Parsecs ?}$$

$$1 \text{ parsec} \simeq 3.10^{16} \text{ m}$$

REDSHIFT

WE WANT TO EXPRESS H AS $H = H(z)$ RECALL!

$$a_0 = a_{TODAY} \equiv 1 \quad ; \quad \frac{a_0}{a(t)} = 1 + z(t)^{*}$$

① RECALL: FOR A MATTER DOMINATED UNIVERSE, ONE FINDS!

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3} \quad ; \quad \text{SO} \quad H = \frac{\dot{a}}{a} = \frac{\frac{2}{3}\left(\frac{t}{t_0}\right)^{-1/3}}{(t/t_0)^{2/3}}$$

REVERSE (a/a_0)

$$\Rightarrow H = \frac{2}{3} \frac{1}{(t/t_0)} = \underbrace{\frac{2}{3t_0}}_{H_0} \frac{1}{(a(t)/a_0)^{3/2}} = \boxed{H_0 (1+z)^{3/2}}$$

② REPEAT FOR $k=0$ WITH ρ_Λ. $\Rightarrow \rho_\Lambda = \rho_{0\Lambda} \Leftarrow$ CONSTANT.

SO $H = \frac{\dot{a}}{a} = \frac{\dot{a}_0}{a_0} = H_0$ ALWAYS SAME EXPANSION RATE.

③ WRITE q AS $q(z, H(z), H'(z))$

$\Rightarrow$ RECALL $\quad q = -\dfrac{a\ddot{a}}{\dot{a}^2}$

FIRST THING IS TO WONDER IF WE CAN EXPRESS $\dfrac{a\ddot{a}}{\dot{a}^2}$ AS A DERIVATIVE: TRY:

$$\left(\frac{a}{\dot{a}}\right)^2 \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \left(\frac{a}{\dot{a}}\right)^2 \frac{\ddot{a}a - \dot{a}^2}{a^2} =$$

$$= \frac{a\ddot{a}}{\dot{a}^2} - 1 =$$

this is

$$= \boxed{-q - 1 \quad = RHS}$$

$$\frac{1}{H^2(t)} d_t H(t) \qquad = \text{CHAIN RULE.}$$

$$\frac{1}{H^2(t)} d_z H(z) \frac{dz}{dt} =$$

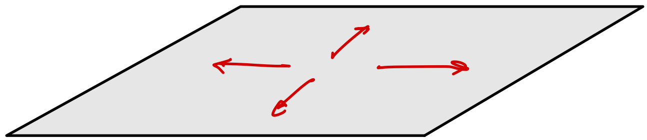
$$d_t\left(\frac{a}{a(t)} - 1\right) = \frac{dz}{dt} = \frac{-a_0\dot{a}}{a^2} = \dot{z}$$

$$\frac{1}{H^2(z)} H'(z) \underbrace{\frac{-\dot{a}}{a}}_{-H(z)} \cdot \underbrace{\frac{a_0}{a}}_{(1+z)}$$

ALL TOGHETHER $\Rightarrow$ $\boxed{\dfrac{H'(z)}{H(z)}(1+z) - 1 = q}$

FLAT EXPANDING UNIVERSE

$K=0$



IT IS EXPANDING W/ DECELERATION "q".

① WE HAVE TO RELATE q WITH w SUCH THAT $q \equiv q(w)$.

WHAT DO WE KNOW? FRIEDMANNS + EQ STATE

• $\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_{tot}$ (1) • $p = \omega\rho$ (4)

• $\frac{2\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G p$ (2)

• $\rho = A \cdot a^{-3(1+\omega)}$ (3)

(1), (3) $\Longrightarrow$ $k=0$ $\rightarrow$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} A \frac{1}{a^{3(1+w)}}$

$\underbrace{\frac{8\pi G}{3} A}_{\tilde{A}}$

(2) $\frac{2\ddot{a}}{a} = -8\pi G w\rho - \frac{8\pi G}{3} A \frac{1}{a^{3(1+w)}} \Rightarrow$

$$\frac{2\ddot{a}}{a} = -\frac{8\pi G}{3} \overbrace{A\, a^{-3(1+w)}}^{\rho}\, 3w - \frac{8\pi G}{3} A\, a^{-3(1+w)}$$

$$= -\hat{A}\, a^{-3(1+w)} \left(1+3w\right)$$

IF MULTIPLY BY $-\frac{a^2}{\dot{a}^2}$

INVERSE OF $\hat{A}\, a^{-3(1+w)}$

$$\Rightarrow -\underbrace{\frac{2\ddot{a}a}{\dot{a}^2}}_{-q} = \bar{A}\, a^{-3(1+w)} (1+3w) \cdot \frac{a^2}{\dot{a}^2}$$

$$q = 1/2\, \hat{A}\, a^{-3(1+w)} (1+3w)\, \bar{A}^{-1}\, a^{3(1+w)}$$

$$\boxed{q = 1/2\, (1+3w)}$$

✓

2 $H \equiv H(a, q, H_0, a_0)$

WE KNOW $H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho}$

$$= \left(\bar{A}\, \frac{1}{a^{3(1+w)}}\right)^{1/2} \Rightarrow \text{BUT } \frac{2q-1}{3} = w$$

$$\Rightarrow H = \frac{\sqrt{\bar{A}}}{a^{1+q}} \; ; \; \text{BUT } \frac{H}{H_0} = \frac{\sqrt{\bar{A}}/a^{1+q}}{\sqrt{\bar{A}}/a_0^{1+q}} \Rightarrow H = H_0 \left(\frac{a_0}{a}\right)^{1+q}$$

③ PROPER DISTANCE? $\quad D \equiv D(z, H_0, q)$

RECALL $\quad D = a_0 r = a_0 \int_0^r dr$

HOW DO WE RELATE $\int dr$ WITH z AND q? TRICK!

WE START FROM $\quad ds^2 = 0 = -dt^2 + a^2(dr^2 + \dots)$

$$\Rightarrow \quad dt = a\,dr \quad = \quad \frac{a_0}{a} dt = a_0\, dr$$

$$\frac{a_0}{a}\frac{dt}{da} da =$$

$$\frac{a_0}{a\dot{a}} da = a_0\, dr$$

$$\frac{1}{H}\frac{a_0}{a^2} da = a_0\, dr$$

BUT $\quad H = H_0 \left(\frac{a_0}{a}\right)^{1+q}.$

ALSO $\Rightarrow \quad \frac{\overset{\text{OBS}}{a_0}}{\underset{\text{emit}}{a^2}} da \Rightarrow -d(1+z) \quad \left(\text{RECALL } \quad 1+z = \frac{a_{obs}}{a_{emit}}\right)$

$$\Rightarrow \quad a_0\, dr = \frac{-d(1+z)}{H_0 (1+z)^{1+q}} \quad \Rightarrow \text{SO}$$

$$D = -\frac{1}{H_0}\int_0^z dz \frac{1}{(1+z^2)^{1+q}} \Rightarrow \boxed{\frac{-1}{qH_0}\left(\frac{1}{(1+z)^q} - 1\right) = D}$$

MATTER DOMINATION

· ASSUME $\rho_m = \rho_{crit.}$ AND $\rho_r = \rho_r$

t WHEN $\rho_r \simeq \rho_m$. ASSUME $H_0 = H_0$.

THIS TYPE OF PROBLEMS ARE EASIER STARTING FROM ENERGY CONSERVATION:

RECALL: $$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\Longrightarrow \rho_i(t) = \rho_{0i}\left(\frac{a(t)}{a(0)}\right)^{-3(1+w)}$$

MATTER $w=0$ $\Longrightarrow$ $\rho_m(t) = \rho_{0m}\left(\frac{a(0)}{a(t)}\right)^3$

RAD $w=1/3$ $\rho_r(t) = \rho_{0r}\left(\frac{a(0)}{a(t)}\right)^4$

SO $\rho_{m0} = \rho_{crit} = \frac{3H_0^2}{8\pi G} \Longrightarrow \left(\frac{a(0)}{a(t)}\right)^3 = \frac{\rho_m(t)}{\rho_{crit.}}$

WHEN $\rho_m(t) = \rho_r(t) \Rightarrow \left(\frac{a(0)}{a(t)}\right)^4 \rho_{0r} = \rho_{crit}\left(\frac{a_0}{a(t)}\right)^3$

$$\frac{\rho_{crit}}{\rho_{or}} = \frac{a(0)}{a(t)} \underset{a(0)=1}{\Longrightarrow} \boxed{a(t) = \rho_{or}/\rho_{crit}} \quad \text{I}$$

YES, BUT WHEN, NOT HOW BIG... RECALL THAT FRIEDMANN EQ GIVE A RELATION BTW SCALE FACTOR AND t. IN FACT.

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{tot} - \frac{k}{a^2}$$

$\rho_{tot} = \rho_{matter} + \rho_{rad}$; BUT AS $\rho_m = \rho_{crit} \Rightarrow$

$\Omega_m \simeq 1 \rightarrow \Omega_r \simeq 0$ AND $k \simeq 0$.

$$\frac{\dot a}{a} = \sqrt{\frac{8\pi G}{3}\rho_{m_0}\left(\frac{a_0}{a}\right)^3}$$

$$\dot a = \sqrt{\frac{8\pi G}{3}\cdot\frac{3H_0^2}{8\pi G}\,\frac{1}{a^3}\cdot a^2}$$

$$\dot a = H_0\frac{1}{\sqrt a} \Rightarrow \int_{a_0}^{a(t_{cross})}\sqrt a\, da = \int_{t_0}^{t_{cross}} H_0\, dt \Rightarrow$$

$$\Rightarrow \frac{2}{3}a^{3/2} \propto H_0 t \Rightarrow a(t) = \left(\frac{3}{2}H_0 t\right)^{2/3} \quad \text{II}$$

I = II

$$t = \frac{2}{3H_0}\left(\frac{\rho_{0r}}{\rho_{crit}}\right)^{3/2} = \boxed{\frac{2}{3H_0}\left(\frac{8\pi G}{3H_0^2}\rho_{0r}\right)^{3/2} = t_{cross}}$$

✓

QUINTESSENCE ELEMENT

- ASSUME TODAY $\equiv t_0$; $\rho_M = \rho_{critic} = \dfrac{3H_0^2}{8\pi G}$;

(a) $k = ??$ $\Rightarrow$ USE FRIEDMANN,

$$\left(\frac{\dot{a}_0}{a_0}\right)^2 = \frac{8\pi G}{3}(\rho_M + \rho_S) - \frac{k}{a_0^2} ;$$

BUT $\rho_M = \dfrac{3H_0^2}{8\pi G}$;

$$H_0^2 = H_0^2 + \frac{8\pi G}{3}\rho_S - \frac{k}{a_0^2} ;$$

SO $k = \dfrac{8\pi G}{3}\rho_S a_0^2$ * ; AS $a_0 > 0$ AND $\dfrac{8\pi G}{3} > 0$ AND

$\rho_S > 0 \Rightarrow \boxed{k > 0}$ CLOSED

(b) EVOLUTION OF UNIVERSE?

PUT $*$ INTO THE FRIEDMANN EQ. (NO) ; NOT FOR TODAY, BUT

FOR ANY OTHER DAY!

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\underbrace{\rho_m\left(\frac{a_0}{a}\right)^3}_{\rho_m \text{ SOME DAY COMPARED TO TODAY.}} + \rho_S\left(\frac{a_0}{a}\right)^?\right) - \underbrace{\frac{8\pi G}{3}\rho_S\left(\frac{a_0}{a}\right)^3}_{*}$$

WHAT IS "?" A QUINTESSENCE HAS $p = w\rho \Rightarrow p = -1/3\,\rho$

SOLVING THE $\partial_t(\rho a^3) = -p\,\partial_t a^3$ WITH $p = w\rho$

ONE GETS :

$$\rho \propto a^{-3(1+w)} \Rightarrow \rho_S \propto a^{-2}\,!$$

SO "?" $= -2$:

SO $\dfrac{\dot{a}^2}{a^2} = \dfrac{8\pi G}{3}\rho\,\dfrac{a_0^3}{a^3} + 0$ (THEY CANCEL)

$$\Rightarrow \int\sqrt{\frac{3}{8\pi G a_0^3}\,a}\;da = \int dt \Rightarrow \boxed{a \propto t^{2/3}}$$

THE SCALE FACTOR DOESNOT

STOP GROWING = EXPANSION

17/MAY/22

$\boxed{\text{WHAT TO SAY}}$

- ANNOUNCE TALK DIMA
- TELL THEM TO DO OLD EXAM (170531)
- TALK ABOUT IMP EQ

FRLW : $ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right)$

FRIEDMAN EQ : $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{tot} - \frac{k}{a^2}$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 = -8\pi G p$$

$\boxed{p = w\rho}$

NULL ENERGY $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \dot\rho + 3(1+w)H\rho = 0$

$w = 0$ (NON RELATIVISTIC MATTER) $\propto 1/a^3$

$w = 1/3$ (γ) $\rightsquigarrow \rho \propto 1/a^4$

$w = -1$ (Λ_{4D}) $\rho \propto$ cte

$\sum \Omega_i = 1$; $\rho_{crit} = \frac{3H_0^2}{8\pi G}$; CONFORMAL $dt = a(\eta)d\eta$

BIG BOUNCE, BIG RIP...

THIS PROBLEM WILL MAKE USE OF THE SAME TECHNOLOGY WE SHOW FOR THE DYNAMICS OF A BLACK HOLE; FROM $\mathcal{L}$ + CONSERVED QUANTITIES WRITE $T+V=E$; BUT INSTEAD OF DOING IT FROM $\mathcal{L}$, DO IT FROM FRIEDMANN EQ AS:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda + \frac{\Omega_k}{a^2}$$

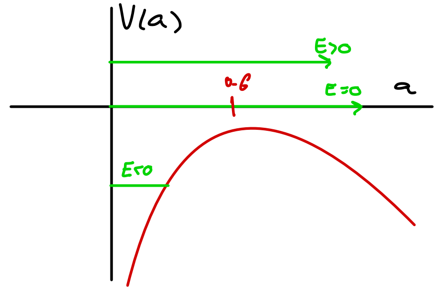
WHERE $\Omega_i = \frac{8\pi G \rho_i}{3H_0^2}$ AND $\Omega_k = -\frac{k}{a_0^2 H_0^2}$

H_0: TODAY; a_0, H_0: TODAY

SO:

$$\underbrace{a^2}_{T} \underbrace{- \left(\frac{\Omega_m}{a} + \frac{\Omega_r}{a^2} + \Omega_\Lambda a^2 \right)}_{-V} = \underbrace{\Omega_k}_{E}$$

FROM POTENTIAL YOU CAN GUESS DYNAMICS:



(a) $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7) \implies \Omega_k = 0$

SO: $V = -\dfrac{0.3}{a} - 0.7a^2$

DOES IT HAVE A METASTABLE POINT?

i.e. $V'(a) = 0$:

$$\frac{\Omega_m}{a^2} - 2\Omega_\Lambda a = 0 \implies a_M = \sqrt[3]{\frac{\Omega_m}{2\Omega_\Lambda}} = \underline{\underline{0.6}}$$

$V(a_M) \simeq -0.75$

$\boxed{a_{TODAY} = 1}$

- IF $E<0 \implies k>0$; SPHERICAL SECTION; THE UNIVERSE COLAPSE
- IF $E=0 \implies k=0$: UNIVERSE EXPANDS: OUR CASE.
- SAME FOR $E>0 = k<0$.

WHAT IS THE ENERGY = − CURVATURE? USE COSMIC SUM RULE.

$$\sum \Omega_i = 1 \implies \Omega_k = 1 - \Omega_m - \Omega_\Lambda$$

$$\Omega_k = E = 1 - 0.3 - 0.7 \implies E = 0$$

- UNIVERSE EXPANDS. AS $E > V(a_M)$

(b) (3, 0.1) COLLAPSE FOR SURE:

$V'(a) = 0 \implies a_m = 2.47$

$V(a_m) = -1.81$

$\Omega_K = 1 - 3 - 0.1 = -2.1 = E$;

AS $E < V(a_m) \implies$ UNIVERSE COLLAPSES

(c) $(0.3, 2) \implies V'(a) = 0 \implies a_m = 0.42$

$V(a_m) = -1.07$

$E = \Omega_K = 1 - 0.3 - 2 = -1.3 \implies$ NO BIG BANG.

UNIVERSAL MERRY GO ROUND.

AS WE ARE DEALING W/ AN UNIVERSE W/ MATTER DOMINATION + $k>0$, WE KNOW THAT IT WILL COLLAPSE DUE TO GRAVITY.

FROM THE 1ST FRIEDMANN EQ:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_M - \frac{k}{a^2}$$

$$\rho_M = \rho_{M0}\left(\frac{\overset{\overset{1}{\shortparallel}}{a_0}}{a}\right)^3 \quad ; \quad \text{SO} \quad \frac{8\pi G}{3}\rho_{M0} = \Omega_M H_0^2 \Big/ k = -\Omega_k H_0^2 \overset{\overset{1}{\shortparallel}}{a_0^2}$$

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_M}{a^3}H_0^2 - \underset{\underset{-k}{\shortparallel}}{\Omega_k} H_0^2}$$

THIS EQUATION IS HARD TO SOLVE IN GLOBAL TIME (GIVE IT A TRY) BUT NOT IN CONFORMAL TIME. THEY ARE RELATED BY:

$$dt = a(\eta)\,d\eta \longrightarrow ds^2 = a^2(\eta)\left(-dt^2 + d\chi^2\right)$$

CONFORMAL RADIUS, DIMENSIONLESS.

THE EQUATION LOOKS AS :

$$\frac{a'^2}{a^4} = \frac{H_0^2 \Omega_M}{a^3} + \frac{\Omega_K H_0^2}{a^2} \quad = \text{MULTIPLY TIMES } a^3 \text{ (CONVENIENCE)}$$

$$\frac{a'}{\sqrt{a}} = H_0 \left(\sqrt{\Omega_M + \Omega_K a} \right)$$

PERFORM CHANGE VARIABLES $\sqrt{x} = a$

$$\frac{2dx}{d\eta} = H_0 \sqrt{\Omega_M + \Omega_K x^2} \implies \eta/2 = \int_0^{\sqrt{a}} \frac{dx}{H_0 \sqrt{}} \implies$$

$$\boxed{a(\eta)\Big|_{K>0} = \frac{\Omega_M}{|\Omega_K|} \sin^2\left(\frac{\sqrt{|\Omega_K|} H_0 \eta}{2} \right)}$$

OBSERVE THAT SCALE FACTOR HAS A MAX OF $\frac{\Omega_M}{|\Omega_K|}$ AND MIN = 0.

By SYMMETRY, $\eta_{COLLAPSE}$ AT:

$$\boxed{\eta_{COL} = \frac{2\pi}{\sqrt{\Omega_K} H_0}} \qquad \eta_{max} = \frac{\eta_{COL}}{2}$$

AS THE PHOTON FOLLOWS NULL GEODESICS:

$$ds^2 = 0 = -a^2(\eta)\,d\eta^2 + a^2(\eta)\,d\chi^2 \Rightarrow \eta = \chi$$

SO $\eta_{COLLAPSE} = \chi_{COLLAPSE}$:

$$\chi_{max} = \eta_{max} = \frac{\chi_{COLLAPSE}}{2} \Rightarrow \text{SO, BACK AND FORTH}$$

$\Rightarrow$ $\boxed{\text{1 LOOP.}}$

- REPEAT SAME COMPUTATION WITH Ω_{rad} INSTEAD OF Ω_M TO SEE:

$$\chi_{COL}\big|_{rad} \propto \pi \Longrightarrow \boxed{\tfrac{1}{2}\ \text{LOOP WHEN RADIATION.}}$$