Electrodynamic Solved Problems

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Contents

1 Introduction

1.1 About these notes

These notes contain a set of selected problems to discuss during the problem solving session of Classical Electrodynamics subject at Uppsala University (Sweden). The order you see in the table of content correspond to chronological order of the lectures for this course. Each set of problems is related to the correponding lectures where that content was discussed during the course (i.e. **L1** = first lecture). The title of each problem statement is linked to its solution. Try first without looking at... Exercises with an **E** in front of them correspond to old exam ones.

In case you find some typo, mistake or section to improve, please send an email to daniel.panizo@... with indications where the issue is[1].

1.2 Recommended Bibliography

- **Classical Electrodynamics**, John David Jackson. You may not like this book at first glance. Neither second, third... but it contains a formal and serious approach to all the topics that are going to be covered during the lectures. It contains important examples and explanations.

- **Introduction to Electrodyamics**, David J. Griffiths. Excellent book for a first approach to many of the concepts in this course. Its level does not cover the one expected for this course, but after reading once[2] you can jump into Jackson.

- **Electromagnetic Field Theory**, Bo Thidé. It does not contain all the material of the course, but it includes several derivations of formulae and a good final appendix with tons of identities and explanations of the mathematical tools.

- **Space and Geometry: An introduction to General Relativity**, Sean Carroll. This is some extra material to read about tensor notation. The first chapter, and part of the second one, cover the properties of the tensorial language we are going to use. This will be useful for the covariant formalism of electrodynamics and Lagrangian manipulation parts of this course.

- **FMM: Exercise Notes**, S.Giri & G. Kälin. Uploaded to Studium. It contains the most useful mathematical methods and examples that show how to use them. Totally recommended to refresh your mathematical manipulation.

[1] Title of the problem, number of equation as reference, etc

[2] Sections, not the whole book.

- **Internet**. As you may know, apart from Social Networks and kitten videos, it contains an enormous amount of resources when used in a proper way.

1.3 Tips to enhance your understanding

Here we offer a set of tips in order to enhance your problem-solving capability.

- Read twice/ thrice/ hundredice the statement of a problem until you really understand what is asking you to solve. You can apply the same principle when reading through sections of books, notes, etc.

- "Pachanguera": Although it is a Spanish word to describe dynamic-noisy-low quality music, it can be also used to describe what a drawing sketch is. It is easier to remember what the problem is asking for if you draw a low quality picture of the set up. You can understand a problem in a better way if you translate to a picture the description given in the statement.

- "Explain yourself": It is nice for your future self[3] and for the people who will correct your exercises/exam if you explain with descriptive sentences the process of your calculations. It gives a context to whoever reads through your problems and help you to stay focus on the final target (solution) you are looking for.

- "Tolle, Lege": Take it, read it. Saint Augustine was wise enough to know that if you do not open and read books, you will not learn. It applies from religion to physics. If you do not understand what you are reading, try first point of these recommendations. Also, you are more than encouraged to ask the Teacher or teacher assistant.

[3] Has it not happened to you that you try to do your exercises again to prepare for the exam and you cannot understand why you calculated something in a particular way?

2 Problems

2.1 Electrostatics (L1, L2)

2.1.1 Conducting ball

A conducting ball of radius R and total charge Q sits in a homogeneous electric field $\vec{E} = E_0\hat{z}$. How does the electric field change by the presence of the ball? (Make an Ansatz of the form $\Phi(r,\theta,\phi) = f_0(r) + f_1(r)\cos\theta$ and motivate it.) Tip: $\hat{Z} = \cos\theta\hat{r} + \sin\theta\hat{\theta}$.

2.1.2 Conducting ball Again

1. A point charge q sits at $\vec{a}$ inside a conducting uncharged sphere that is earthed with radius $R(|\vec{a}| < R)$.Compute the potential and the electric field inside the sphere using the method of mirror charges. Compute also the induced charge density on the surface of the sphere and show that the total charge on the surface is $-q$. What does the Gauss theorem say about the electrical field outside the sphere?

2. Do the same analysis with the change that the sphere is isolated and uncharged. Tip: Determine the electric field outside the sphere with the new b.c.

3. Follow again the same procedure as b for a sphere that is isolated and with charge Q.

2.1.3 The Capacitance of an off-centered Capacitor

A spherical conducting shell centered at the origin has radius R_1 and is maintained at potential V_1. A second spherical conducting shell maintained at potential V_2 has radius $R_2 > R_1$ but is centered at the point $s\hat{\mathbf{z}}$ where $s \ll R_1$.

1. To lowest order in s, show that the charge density induced on the surface of the inner shell is

$$\sigma(\theta) = \epsilon_0 \frac{R_1 R_2 (V_2 - V_1)}{R_2 - R_1} \left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3} \cos\theta \right]. \qquad (2.1.1)$$

   Hint: Show first that the boundary of the outer shell is $r_2 \approx R_2 + s\cos\theta$.

2. To lowest order in s, show that the force exerted on the inner shell is:

$$\mathbf{F} = \int dS \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int^{\pi} d\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0}\cos\theta = -\frac{Q^2}{4\pi\epsilon_0}\frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3}. \qquad (2.1.2)$$

2.1.4 Spherical cavity and spherical functions

Consider a sphere of radius a where the surface of the upper hemisphere has a potential $+\Phi_0$ and the surface of the lower hemisphere has a potential $-\Phi_0$. In this case the Green Function is given by:

$$G(r, r') = \frac{1}{\left|\vec{r} - \vec{r'}\right|} - \frac{a}{r'\left|\vec{r} - \frac{a^2}{r'^2}\vec{r'}\right|}, \qquad (2.1.3)$$

where $\vec{r'}$ refers to a unit source outside the sphere and $\vec{r}$ to the point where the potential is evaluated.

1. Using the expression for the expansion of $\frac{1}{\left|\vec{r}-\vec{r'}\right|}$ in the appropriate basis show that the Green's function can be written as

$$G(r, r') = 4\pi \sum_{l,m} \frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right] Y^*_{l,m}(\theta', \phi') \, Y_{l,m}(\theta, \phi), \qquad (2.1.4)$$

2. Using Dirichlet boundary conditions, show that the potential outside the sphere has fol- lowing the expansion.

$$\Phi(r, \theta, \phi) = \sum_{lm} \frac{l+1}{a^2(2l+1)}\left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta, \phi) \int \Phi_0(\theta', \phi') \, Y^*_{l,m}(\theta', \phi') \, d\Sigma', \quad (2.1.5)$$

which tends to 0 as $r \to \infty$.

2.1.5 Green's function between concentric spheres

Consider the green's function for Neumann b.c. in the volume V between two concentric spheres between $r = a$ and $r = b, a < b$. We write the potential as

$$\Phi(x) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x') \, G(x, x') \, d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'} G da', \qquad (2.1.6)$$

where S is the surface of the boundary. This implies that the b.c. for the Green's function is given by:

$$\frac{\partial}{\partial n'} G\left(x, x'\right) = -\frac{4\pi}{S}, \tag{2.1.7}$$

or x' in S. Expanding the Green's function in spherical harmonics we get:

$$G\left(x, x'\right) = \sum_{l=0}^{\infty} g_l\left(r, r'\right) P_l(\cos\gamma), \tag{2.1.8}$$

where $g_l\left(r, r'\right) = \frac{r_l^l}{r_>^{l+1}} + f_l\left(r, r'\right)$, and γ is the angle between the vector x and x'.

Also here one can prove that $P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_m Y_{l,m}^*\left(\theta', \phi'\right) Y_{l,m}(\theta, \phi)$.

1. Show for $l > 0$ that the Green's function takes the symmetric form:

$$g_l\left(r, r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l}\left(r r'\right)^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{\left(r r'\right)^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right] \tag{2.1.9}$$

2. Use the Green's function that you found in the situation that you have a normal electric field $E_r = -E_0 \cos\theta$ at $r = b$ and $E_r = 0$ at $r = a$. Show that the potential inside V is

$$\Phi(x) = E_0 \frac{r\cos\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right), \tag{2.1.10}$$

where $p = \frac{a}{b}$. Find also for the electric field that:

$$E_r(r, \theta) = -E_0 \frac{\cos\theta}{1 - p^3}\left(1 + \frac{a^3}{r^3}\right), \quad E_\theta(r, \theta) = E_0 \frac{\sin\theta}{1 - p^3}\left(1 + \frac{a^3}{2r^3}\right). \tag{2.1.11}$$

2.2 Multipoles (L3)

2.2.1 Spherical Multiple Moment

Consider the system where you have point charges $+q$ at $(a, 0, 0)$ and $(0, a, 0)$ and charges $-q$ at $(-a, 0, 0)$ and $(0, -a, 0)$. Derive the spherical multiple moment $q_{l,m}$ and write down the first two non vanishing terms. Express the charge density in spherical coordinates and check that the integral over these densities produce the appropriate total charge.

2.2.2 Multiple Moments in Cartesian Coordinates

1. Prove that Q_{ij} is traceless.

2. Assume that $q, \vec{p}, Q_{ij}$ are in a specific coordinate system. Now find the new quantities in a coordinate system which is related to the previous one by an $\vec{R}$ displacement. Assume now that you have charges q at $(0, a, 0)$ and $(0, 0, a)$ and charge $-q$ at $(a, 0, 0)$

3. Find $q, \vec{p}, Q_{ij}$ and check that the later one is traceless.

4. Can you find a coordinate system such that $\vec{p}' = 0$? If yes what is the displacement vector $\vec{R}$?

2.2.3 Exterior Multipoles for a Specified Potential on a Sphere

Let $\Phi(R, \theta, \phi)$ be specified values of the electrostatic potential on the surface of a sphere. Show that the general form of an exterior, spherical multipole expansion implies that,

$$\Phi[\vec{r}] = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{R}{r}\right)^{l+1} Y_{l,m}[\Omega] \int d\Omega' \Phi[R, \Omega'] Y^*_{l',m'}[\Omega'] \tag{2.2.1}$$

For $r > R$. Given the previous potential expression, imagine the eight octants of a spherical shell which are maintained at alternating electrostatic potentials $\pm V$ as shown below in the following picture:

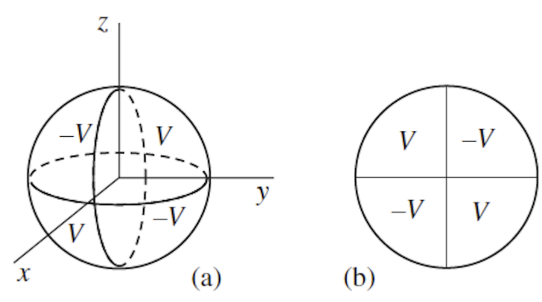


Figure 1: Potential distribution across the octants.

Where view a is in perspective and b is looking down the z axis from above. Use the results from previous section to find the asymptotic ($r \to \infty$) form of the potential produced by this shell configuration.

2.2.4 Radiating Fidget Spinner

Three identical point charges q are at the corners of an imaginary equilateral triangle that lies in the $x - y$ plane. The charges rotate with constant angular velocity ω around the z -axis, which passes through the center of the triangle. Find the angular distribution of electric dipole, magnetic dipole, and electric quadrupole radiation (treated separately) produced by this source.

2.3 Macroscopic Media (L3, L4)

2.3.1 A Conducting Sphere at a Dielectric Boundary

A conducting sphere with radius R and charge Q sits at the origin of coordinates. The space outside the sphere above the $z = 0$ plane has dielectric constant κ_1. The space outside the sphere below the $z = 0$ plane has dielectric constant κ_2.

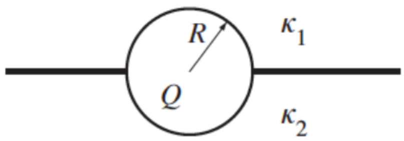


Figure 2: Dielectric distribution around the sphere.

1. Find the potential everywhere outside the conductor.

2. Find the distributions of free charge and polarization charge wherever they may be.

2.3.2 Polarization by Superposition

Two spheres with radius R have uniform but equal and opposite charge densities $\pm\rho$. The centers of the two spheres fail to coincide by an infinitesimal displacement vector δ. Show by direct superposition that the electric field produced by the spheres is identical to the electric field produced by a sphere with a suitably chosen uniform polarization **P**.

2.3.3 The Field at the Center of a Polarized Cube

A cube is polarized uniformly parallel to one of its edges. Show that the electric field at the center of the cube is $\mathbf{E}(0) = -\mathbf{P}/3\epsilon_0$. Compare with $\mathbf{E}(0)$ for a uniformly polarized sphere. Hint: Recall the definition of solid angle.

2.3.4 E and D for an Annular Dielectric

1. The entire volume between two concentric spherical shells is filled with a material with uniform polarization $\mathbf{P}$. Find $\mathbf{E}(\mathbf{r})$ everywhere.

2. The entire volume inside a sphere of radius R is filled with polarized matter. Find $\mathbf{D}(\mathbf{r})$ everywhere if $\mathbf{P} = P\hat{\mathbf{r}}/r^2$.

2.3.5 E: A Charge and A Conducting Sphere

1. A charge q is placed at a distance d away from the center of a conducting sphere of radius $a < d$. Let the potential at infinity and on the surface of the sphere be 0. Using the method of images find the total charge induced on the surface of the sphere.

2. Suppose the conducting sphere and the charge q are as above but the potential on the surface of the sphere is $V \neq 0$ (the potential at infinity is 0). Find the total charge on the surface of the sphere (hint: you need to place a second "image charge" at the center of the sphere).

3. Now consider a different situation. There are two conducting spheres of radius a whose centres are at a distance d that is much greater than a. The potential at infinity is 0. One of the spheres is kept at a potential V and the other at $-V$. Because $a \ll d$ when discussing the fields near one of the spheres you can approximate the other sphere as a single point charge located at its center. Using this approximation find the total charge on the surface of each of the spheres.

4. Finally imagine that the space in between the two spheres is filled with a medium of conductivity σ so that, in the presence of an electric field, there will be a current density $\vec{J} = \sigma\vec{E}$. Using Gauss's law find the total current I flowing between the two spheres. (Note: ignore the effects of any $\vec{B}$ produced by the moving charges). Compute the effective resistance of the circuit $R = \frac{2V}{I}$ as a function of a and d. What happens to R as $d \to \infty$. What happens to R as $a \to 0$?

5. (For a bonus point) Can you give a qualitative reason for the behavior of R found above? (Hint: think of resistors in series and parallel).

2.3.6 E: Critical strain

A parallel plate capacitor is made of two identical parallel conducting plates of area A. One plate carries a charge $+q$ and the other a charge $-q$. The capacitor is filled with a dielectric medium with permittivity ϵ. The distance between the two plates d is variable because the dielectric is elastic. The elastic energy stored in the dielectric is:

$$U_{\text{el}} = \frac{1}{2} k\,(d - d_0)^2 \,. \tag{2.3.1}$$

where d_0 and k are constants.

1. Find the separation of the plates at equilibrium $d(q)$.

2. Find and plot the potential difference between the plates at equilibrium $V(q)$ as a function of q. Interpret the result.

2.4 Light and Polarisation (L5, L6)

2.4.1 Elliptic Polarisation Wave

Assume electromagnetic wave $\vec{E}(x,t)$ and the magnetic part of it that will not contribute in the exercise. The propagation vector is in the z direction $\vec{k} = k\hat{z}$ and the wave has the following form

$$E_x(\vec{x},t) = A\cos(kz - \omega t), \tag{2.4.1a}$$
$$E_y(\vec{x},t) = B\cos(kz - \omega t + \phi). \tag{2.4.1b}$$

1. Show that the vector $\vec{E}(0,t)$ parametrizes an ellipse. Note that this vector describe the polarization. For which values of A, B and ϕ the polarization parametrizes a circle? Tip: The ellipse equation is of the form $ax^2 + 2bxy + cy^2 + f = 0$.

2. Show for general A and B that the wave can be written as a superposition of two opposite circular polarized waves

$$\vec{E}(\vec{x},t) = \operatorname{Re}\left(\vec{E}_+(z,t) + \vec{E}_+(z,t)\right) \tag{2.4.2}$$

where $\vec{E}_\pm(z,t) = A_\pm \epsilon_\pm e^{i(kz-\omega t)}$. Here we have that $A_\pm$ are constants that need to be found and $\epsilon_\pm = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$.

2.4.2 A Sandwich of Light

Assume two half planes made out of a homogeneous isotropic, non magnetic, loss-free, dielectric medium with refraction index n. The two planes are separated by vacuum and they are d distance away from each-other.

A wave is propagated from the below hitting the first surface of the medium with vacuum with angle α. The wave has frequency ω.

Consider the two cases where the propagation is perpendicular to the plane of incident. Describe the phenomenon and find how much of the wave was transmitted or reflected (energy/time).

2.4.3 Faraday Rotation During Propagation

For propagation along the z -axis, a medium supports left circular polarization with index of refraction n_L and right circular polarization with index of refraction n_R. If a plane wave propagating through this medium has $\mathbf{E}(z=0,t)=\hat{\mathbf{x}}E\exp(-i\omega t)$, find the values of z where the wave is linearly polarized along the y -axis.

2.4.4 Charged Particle Motion in a Circularly Polarized Plane Wave

A particle with charge q and mass m interacts with a circularly polarized plane wave in vacuum. The electric field of the wave is $\mathbf{E}(z,t)=\mathrm{Re}\left\{(\hat{\mathbf{x}}+i\hat{\mathbf{y}})E_0\exp[i(kz-\omega t)]\right\}$.

1. Let $v_\pm = v_x \pm i v_y$ and $\Omega = 2qE_0/mc$. Show that the equations of motion for the components of the particle's velocity v can be written

$$\frac{dv_z}{dt} = \frac{1}{2}\Omega\left\{v_+ e^{+i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right\} \tag{2.4.3a}$$

$$\frac{dv_\pm}{dt} = \Omega\,(c - v_z)\,e^{\mp i(kz-\omega t)} \tag{2.4.3b}$$

2. Let $\ell_\pm = v_\pm e^{\pm i(kz-\omega t)} \pm ic\Omega\omega$ and show that

$$\frac{dv_z}{dt} = \frac{1}{2}\Omega\,(\ell_+ + \ell_-) = i\frac{\Omega}{2\omega}\frac{d}{dt}\,(\ell_+ - \ell_-) \tag{2.4.4}$$

3. Let K be the constant of the motion defined by the two $\dot{v}_z$ equations above. Differentiate the equations in part (a) and establish that

$$\frac{d^2 v_z}{dt^2} + \left[\Omega^2 + \omega^2\right] v_z = \omega^2 K \tag{2.4.5}$$

Use the initial conditions $v(0) = 0$ and $v_z'(0) = 0$ to evaluate K and solve for $v_z(t)$. Describe the nature of the particle acceleration in the z -direction.

2.4.5 E: A Wave and Some Boundary Conditions

Consider an electromagnetic wave propagating in the vacuum in the half-space $x_3 \geq 0$.

$$\vec{E}_i(\vec{x}, t) = \vec{E}_0 e^{\mathrm{i}\vec{k}\cdot\vec{x} - \mathrm{i}\omega t}, \tag{2.4.6a}$$

$$\vec{B}_i(\vec{x}, t) = \frac{\hat{k}}{c} \times \vec{E}, \tag{2.4.6b}$$

where $\vec{E}_i$ satisfies $\vec{k}\cdot\vec{E}_i = 0$ and the components of $\vec{k}$ are real. The frequency satisfies $\omega^2 = c^2\vec{k}\cdot\vec{k}$.

1. Suppose this wave is incident on a perfectly conducting plane placed at $x_3 = 0$. Let the plane of incidence be formed by $\vec{k}$ and $\hat{x}_3$. Write down an expression for the electric and magnetic fields for the reflected wave $\vec{E}_r$ and $\vec{B}_r$. (Consider separately the case where $\vec{E}_r$ and $\vec{E}_i$ are both perpendicular to the plane of incidence and the case where they are both contained in it.)

2. Now suppose there is a second conducting plane located at $x_3 = d > 0$. Derive what are the conditions on $\vec{k}$, such that in the region $0 < x_3 < d$ the electric and magnetic fields are given by:

$$\vec{E} = \vec{E}_i + \vec{E}_r, \quad \vec{B} = \vec{B}_i + \vec{B}_r, \tag{2.4.7}$$

   where the incident and reflected fields are those found above.

3. Suppose now that the two conducting planes are orthogonal to each other. One is placed at $x_3 = 0$ and the other at $x_1 = 0$. How many plane-waves do you need generically to satisfy the Maxwell equations (with the appropriate boundary conditions) in the region $x_1 > 0, \quad -\infty < x_2 < +\infty, \quad x_3 > 0$? Write down the electric and magnetic fields for one such solution.

2.4.6 E: Waving at the Properties of a Wave

Let $\vec{E} = \hat{y}E_0 e^{i(hz-\omega t)-\kappa x}$ be the electric field of a wave propagating in vacuum. The parameters E_0, h, ω, κ are real.

1. What is the magnetic field of the wave?

2. Use the wave equation for $\vec{E}$ to determine a relation between h, κ and ω.

3. Compute the time averaged Poynting vector.

2.5 Waveguides and Cavities (L7 , L8)

2.5.1 Electromagnetic Crosswalk

Imagine two electromagnetic beams intersecting at right angles. $(\bar{E_H}, \bar{B_H})$ (moving in the horizontal direction) propagates in the $+x$ axis. $(\bar{E_V}, \bar{B_V})$ (Vertical direction) propagates in the $+y$ direction. For simplicity, each beam is taken as a pure plane wave cut of transversely so its cross section is a perfect square of area λ^2 (Here λ stands for the "space" each beam occupy). The fields are given by:

$$\vec{E}_H = -E_0 e^{i(kx-\omega t)}\hat{z} \tag{2.5.1a}$$
$$c\vec{B}_H = E_0 e^{i(kx-\omega t)}\hat{y} \tag{2.5.1b}$$
$$\vec{E}_V = E_0 e^{i(ky-\omega t)}\hat{z} \tag{2.5.1c}$$
$$c\vec{B}_V = E_0 e^{i(ky-\omega t)}\hat{x} \tag{2.5.1d}$$

Where $|x| = |y| = |z| < \lambda/2$. The beams overlap in a cube centered at the origin where the total fields are given by a linear combination of vertical and horizontal ones.

1. Calculate the time-averaged energy density $\langle u_{EM}(\bar{r})\rangle$ for the horizontal beam, the vertical beam and the total field in the overlap region. Show that the least of these takes its minimum value on the plane $x = y$. Compute $\bar{E}$ and $\bar{B}$ on this plane.

2. Calculate the time-averaged Poynting vector $\langle S(\bar{r})\rangle$ for the H beam, the V beam and the total field as in previous part. Try to make a sketch of $\langle S(x, y)\rangle$ everywhere the fields are defined.

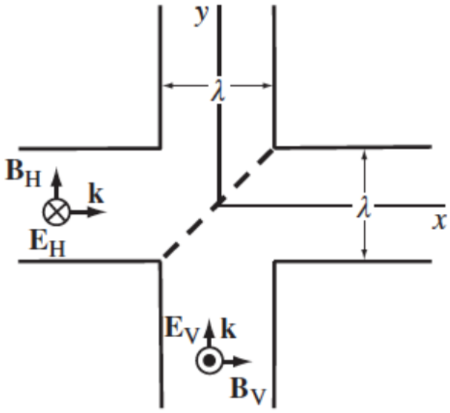


Figure 3: A sketch representation of the crossing beams and their components.

2.5.2 Waveguide Discontinuity

Two rectangular waveguides with different major sides ($a_1 < a_2$) along the x-axis and equal minor sides ($b_1 = b_2$) along the y-axis[4] are joined in the $z = 0$ plane ($x = y = 0$). The first region (a_1) propagates a $TE_{1,0}$ mode in the $+z$-direction towards the second region (a_2). Find the amplitude of some excited modes in the second region. Check also the limit where $a_1 = a_2$[5].

2.5.3 Guess Who? (Wavefilter Edition)

The figure below shows two circular conducting tubes in cross section. Each tube has a thin metal screen inserted at one point along its length. One screen takes the form of metal wires bent into concentric circles. The other takes the form of metal wires arranged like the spokes of a wheel. One of these tubes transmits only a low-frequency TE waveguide mode down the tube. The other transmits only a low-frequency TM waveguide mode down the tube. Explain which tube is which and why, using the fact that the fields of a general waveguide satisfy $\nabla \times \mathbf{E}_\perp = i\omega B_z \hat{z}$.

2.5.4 An Electromagnetic Bat in a Resonant Cavity

The two-dimensional vectors $\mathbf{k}_m$ shown below are inclined at angles $\theta_m = m\pi/3$ with respect to the positive x -axis. The vectors share a common magnitud $|\mathbf{k}_m| = k$. Superpose six waves with alternating amplitudes to form the scalar function

[4] I rotate the axis in my solution, but the result should be the same.

[5] To find the modes and limits, consider that the remaining open space at $x = y = 0$ between waveguides is closed by a perfect conductor, so modes cannot scape from our set up.

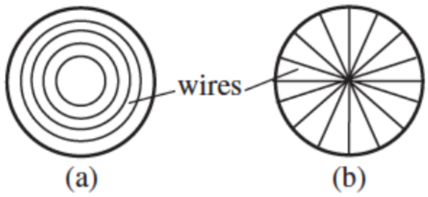


Figure 4: Both described wavefilters.

$$\psi(x,y,t) = \sum_{m=0}^{5} (-1)^k \sin(\mathbf{k}_i \cdot \mathbf{r} - ckt) \tag{2.5.2}$$

Draw the outline of a two-dimensional resonant cavity which supports a TM mode built from $\psi(x,y,t)$.

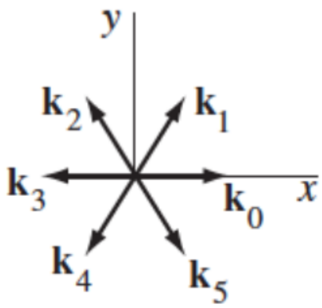


Figure 5: The vectorial distribution of the six waves.

2.5.5 Cutting off the Modes

Transverse electric and magnetic waves are propagated along a hollow, right, circular cylinder with inner radius R and conductivity σ. Find the cutoff frequencies of the various TE and TM modes. Determine numerically the lowest cutoff frequency (dominant mode) in terms of the tube radius and the ratio of cutoff frequencies of the next four higher modes to that of the dominant mode. For this part, assume that the conductivity of the cylinder is infinite.

2.5.6 E: Rectangular Waveguide and its Modes

Consider a waveguide whose section in the x-y plane is a rectangle with sides of length a and b (see figure). The waveguide walls are perfect conductors. The inside of the waveguide can be considered to be the vacuum.

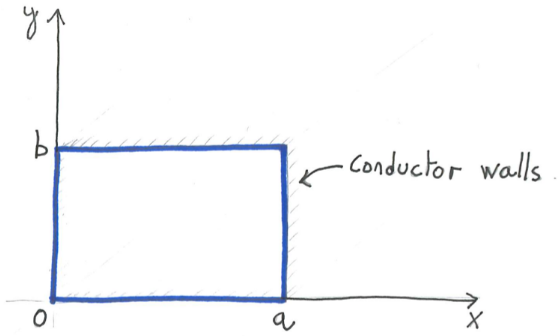


Figure 6: A sketch picture of the waveguide's section.

1. What are the boundary conditions that the electric $\vec{E}$ and magnetic $\vec{B}$ fields need to satisfy at the surface of a perfect conductor?

2. Consider a function $\psi(x, y)$ that satisfies the equation

$$\left(\partial_x^2 + \partial_y^2\right)\psi(x, y) + \gamma^2\psi(x, y) = 0. \tag{2.5.3}$$

in the interior of the rectangle for some $\gamma > 0$. The cutoff frequencies of TE and TM modes for the waveguide are obtained determining the possible values of $\gamma > 0$ in the equation above provided that the function $\psi(x, y)$ satisfies certain boundary conditions at the walls of the waveguide. For TM modes it must be that

$$\psi\big|_{\text{wall}} = 0, \tag{2.5.4}$$

while for TE modes,

$$\left.\frac{\partial\psi}{\partial n}\right|_{\text{wall}} = 0. \tag{2.5.5}$$

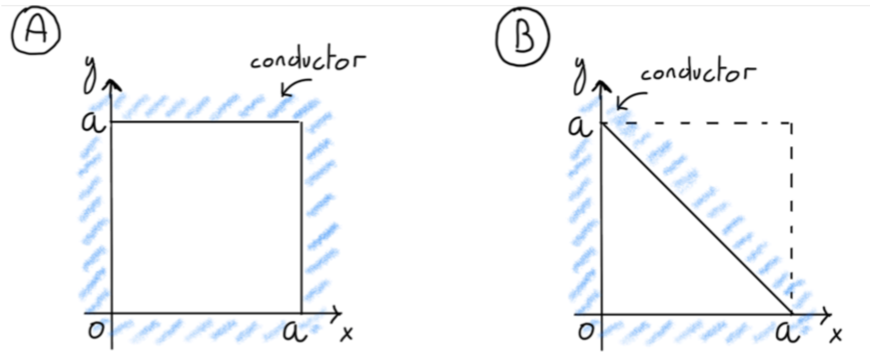
Where $\left.\frac{\partial\psi}{\partial n}\right|_{\text{wall}}$ is the derivative in the direction perpendicular to the wall. The cutoff frequencies are then given by $\omega = c\gamma$.

   (a) For TM modes what is the smallest cut-off frequency?

(b) For TE modes what is the smallest cut-toff frequency?

2.5.7 E: Mirror mirror on the wall...

Consider a waveguide whose section in the x-y plane is a square with sides of length a (see figure A below). The waveguide walls are perfect conductors. The inside of the waveguide can be considered to be the vacuum.



1. What are the boundary conditions that the electric $\vec{E}$ and magnetic $\vec{B}$ fields need to satisfy at the surface of a perfect conductor?

2. Find the TM and TE modes for this wave-guide. For each mode display $\vec{E} \cdot \hat{z}$ for the TM modes and $\vec{B} \cdot \hat{z}$ for the TE modes. Also find the cutoff frequency for every mode.

3. Certain distinct modes have the same cutoff frequency. Why? By taking appropriate linear combinations of the modes sharing the same cutoff frequency construct TM and TE modes for a waveguide whose section is a right isosceles triangle with catheti (short sides) of length a (see figure B above). Show explicitly $\vec{E} \cdot \hat{z}$ for the TM modes and $\vec{B} \cdot \hat{z}$ for the TE modes.

2.6 Radiation and Scattering (L9, L10)

2.6.1 Electric Dipole Radiation

Imagine two tiny metal spheres at distance d from each other connected by a wire, where at time t, the one sphere carries a charge $q(t) = q_0 \cos(\omega t)$ while the other sphere is given by $-q(t)$.

1. Calculate the electric potential far away from the dipole. Use $d << r$ and $d << \frac{c}{\omega}$

2. Take the limit of $\omega \to 0$. What do you expect?

3. Now look at the case where also $r >> \frac{c}{\omega}$, that is, when we are interested in large distances from the source in comparison to the wavelength. How does the expression for the potential simplify in this case?

4. Obtain an expression for the vector potential in the limit $d << r$ and $d << \frac{c}{\omega}$.

5. Calculate the resulting electric and magnetic fields in the same limit with also $r >> \frac{c}{\omega}$.

2.6.2 Metallic Shells

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. an alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V \cos \omega t$. In the long-wavelength limit, find the radiation field, the angular distribution of radiated power and the total radiated power from the sphere.

2.6.3 Electrostatic Potential from a Dipole

Consider a dipole that has distance $\vec{x}'$ and a point P at distance $\vec{x}$ far away from the dipole. Considering the general expression for the potential without boundary conditions show that at large distances from the charge distribution the potential can be approximated by using the electric dipole moment in first order. Then calculate the potential in the case where the dipole is formed by two charges q^+ and q^- with distance d between them.

2.6.4 Radiation Interference

Let the origin of coordinates be centered on a compact, time-harmonic source of electromagnetic radiation. The time-averaged power radiated into a differential element of solid angle $d\Omega$ centered on an observation point $\mathbf{r}$ has the form

$$\frac{dP}{d\Omega} \propto |\hat{\mathbf{r}} \times \alpha| \tag{2.6.1}$$

The vector $\alpha = \mathbf{p}_0$ if the source has a time-dependent electric dipole moment $\mathbf{p}(t) = \mathbf{p}_0 \cos \omega t$. The vector $\boldsymbol{\alpha} = \mathbf{m}_0 \times \hat{\mathbf{r}}$ if the source has a time-dependent magnetic dipole moment $\mathbf{m}(t) = \mathbf{m}_0 \cos \omega t$. For this problem, consider a source where $\mathbf{p}(t)$ and $\mathbf{m}(t)$ are present simultaneously.

1. Show that the time-averaged angular distribution of power generally exhibits interference between the two types of dipole radiation. Under what conditions is there no interference?

2. Show that the time-averaged total power emitted by the source does not exhibit interference.

2.6.5 Sinusoidal thin Antenna

A thin linear antenna of length d is excited in such a way that the sinusoidal current makes a full wavelength of oscillation.

1. Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.

2. Determine the total power radiated and find a numerical value for the radiation resistance.

3. Calculate the multipole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly.

2.6.6 Scattering in Solid Sphere

A solid uniform sphere of radius R and conductivity σ acts as a scatterer of a plane-wave beam of unpolarized radiation of frequency ω, with $\omega R/c << 1$. The conductivity is large enough that the skin depth δ is small compared to R.

1. Justify and use a magnetostatic scalar potential to determine the magnetic field around the sphere, assuming the conductivity is infinite.

2. determine the absorption cross section of the sphere. Tip: The power loss from a waveguide is $\frac{P_{\text{loss}}}{da} = \frac{1}{2\sigma\delta}|\hat{n} \times \vec{H}|^2$.

2.6.7 Aperture (Science)

The aperture or apertures in a perfectly conducting plane screen can be viewed as the location of effective sources that produce radiation (the diffracted fields). An aperture whose dimensions are small compared with a wavelength acts as a source of dipole radiation with the contributions of other multipoles being negligible.

1. Show that the effective electric and magnetic dipole moments can be expressed in terms of integrals of the tangential electric field in the aperture as follows:

$$\vec{p} = \epsilon \hat{n} \int \left(\vec{x} \cdot \vec{E}_{tan}\right) da, \tag{2.6.2a}$$

$$\vec{m} = \frac{2}{i\omega\mu} \int \left(\hat{n} \times \vec{E}_{tan}\right) da. \tag{2.6.2b}$$

where $\vec{E}_{\rm tan}$ is the exact tangential electric field in the aperture, $\hat{n}$ is the normal to the plane screen, directed into the region of interest, and the integration is over the area of the openings.

2. Show that the expression for the magnetic moment can be transformed into

$$\vec{m} = \frac{2}{\mu} \int \vec{x}(\hat{n} \cdot \vec{B}) da. \tag{2.6.3}$$

2.6.8 Born Scattering from a Dielectric Cube

A plane wave $\mathbf{E}_0 \exp\left[i\left(\mathbf{k}_0 \cdot \mathbf{r} - \omega t\right)\right]$ scatters from a dielectric cube with volume $V = a^3$ and electric susceptibility $\chi \ll 1$. Two cube edges align with $\mathbf{k}_0$ and $\mathbf{E}_0$.

1. Calculate the differential scattering cross section in the Born approximation.

2. Show that $\sigma_{\rm Born} \approx \frac{1}{4} k^2 a^4 \chi^2$ when $ka \gg 1$. Hint: The near-forward direction dominates the scattering when $ka \gg 1$

3. The weak scattering assumed by the Born approximation implies that

$$|\mathbf{E}_{\rm rad}| \,/\, |\mathbf{E}_0| \ll 1, \tag{2.6.4}$$

for all $\mathbf{q}$, even when $r \approx a$. Deduce from this that the $ka \gg 1$ result of part (b) is valid only when $\sigma_{\rm Born} \ll \chi a^2$.

2.6.9 E: Two Antennas Sitting Together

A circular loop of radius a made of conducting wire is centred at the origin and lies in the $x_3 = 0$ plane. Let be the polar angle in the $x_3 = 0$ plane (i.e. figure). The wire carries a current oscillating at frequency

$$\vec{I} = I_0 \hat{\phi} e^{-i\omega t}, \tag{2.6.5}$$

with I_0 real. There is also a small antenna wire of length $2a$ along $\hat{x}_3$ centered at the origin as in figure. An oscillating current is fed into the antenna at its midpoint so that, away from the midpoint, the wire carries a linear charge density

$$\lambda = i\lambda_0 e^{-i\omega t}, \quad \text{for} \quad 0 < x_3 < a, \quad \lambda = -i\lambda_0 e^{-i\omega t} + c.c, \quad \text{for} \quad -a < x_3 < 0. \quad (2.6.6)$$

Where λ_0 is real.

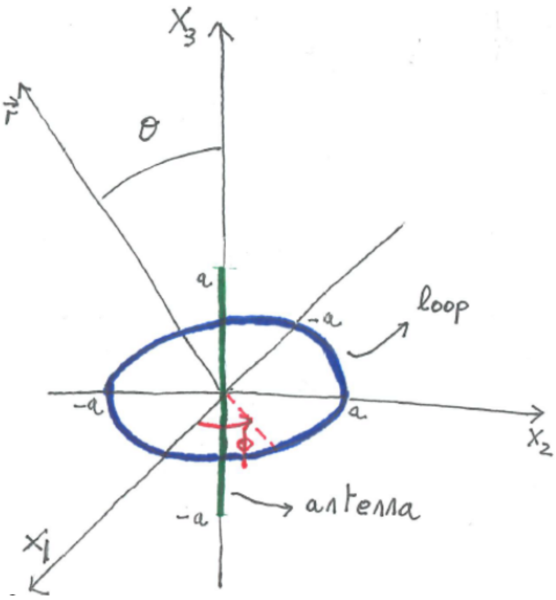


Figure 7: The two described antennas.

1. Find the electric dipole moment $\vec{p}(\omega)$ and the magnetic dipole moment $\vec{m}(\omega)$ at fre- quency ω due to the antenna and to the wire loop.
2. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge due to the dipole moments above in the radiation zone (that is $|\vec{r}| \gg \frac{c}{\omega}$).
3. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.
4. Determine the power emitted per unit solid angle by the antenna and loop in the radiation zone. Write the answer as a function of the angle θ between $\hat{x}_3$ and $\hat{r}$.

2.6.10 E: One... Err, Two Antennas

Consider a small antenna wire of length $2a$ along $\hat{x}_3$. Let the center of the wire be at the origin. A current oscillating at frequency ω is fed into the antenna at its midpoint so that away from the midpoint, the wire carries a linear charge density

$$\lambda = \lambda_0 e^{-i\omega t} \text{ for } 0 < x_3 < a, \quad \lambda = -\lambda_0 e^{-i\omega t} \quad \text{for} \quad -a < x_3 < 0, \tag{2.6.7}$$

Where λ_0 is real.

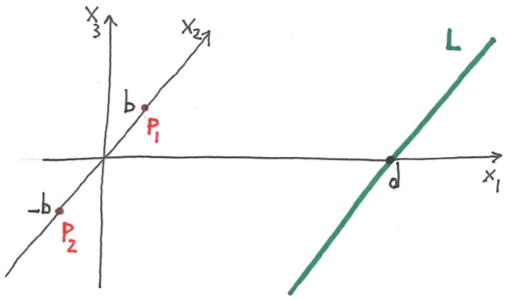


Figure 8: The aforamentioned antennas.

1. Find the electric dipole moment at frequency ω of the antenna $\vec{p}(\omega)$.
2. Determine the current $\vec{I}(x_3)$ flowing along the wire.
3. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge due to the antenna in the radiation zone (that) is $|\vec{r}| \gg \frac{c}{\omega}$).
4. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.
5. Now consider placing two antennas identical to the one above at the two points (see figure)

$$P1 : (x_1 = 0, x_2 = b, x_3 = 0) \text{ and } \mathrm{P2} : (\mathrm{x_1 = 0, x_2 = -b, x_3 = 0}) \,. \tag{2.6.8}$$

The two antennas are pointing along $\hat{x}_3$ and they are oscillating in phase.

(a) Let $b = \frac{5\pi c}{\omega}$. Determine the electric field $\vec{E}(x_2)$ along the line L (see figure) located at $x_3 = 0, x_1 = d$. Assume that $d \gg b$

(b) Does the electric field you found above vanish somewhere along the line L? If so where? Explain your result.

2.6.11 E : Who bent my Antenna?

An antenna is made of a circular conducting wire loop of radius a centered at the origin. It lies in the $x = 0$ plane. Let $-\pi < \alpha \leq \pi$ be the polar angle in the $x = 0$ plane (see figure at the top of next page). There is a gap in the wire at $\alpha = \pi$ so no current can flow across. The antenna is fed an RF signal at $\alpha = 0$ so that the wire carries a current oscillating at frequency ω

$$\vec{I} = I_0(\pi - |\alpha|)\hat{\alpha}e^{-i\omega t}, \quad -\pi < \alpha < \pi, \tag{2.6.9}$$

with I_0 real.

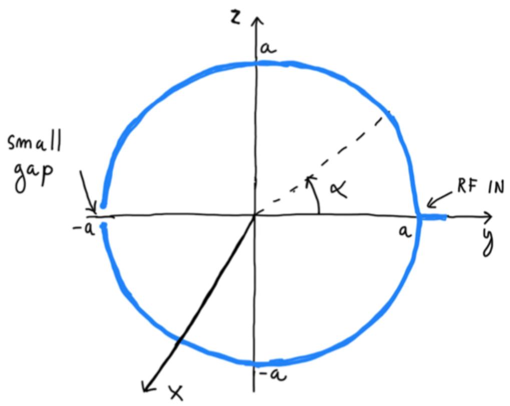


Figure 9: Who bent it?

1. Find the electric dipole moment $\vec{p}(\omega)$ and the magnetic dipole moment $\vec{m}(\omega)$ at frequency ω of the wire loop.

2. Work in the approximation that $\frac{c}{\omega} \gg a$ so that a multipole expansion is meaningful. Determine the vector potential $\vec{A}(\vec{r},\omega)$ in Lorentz gauge in the radiation zone (that is $|\vec{r}| \gg \frac{c}{\omega}$) due to the dipole moments above.

3. In the same approximation write down the electric and magnetic fields $\vec{E}(\vec{r},\omega)$ and $\vec{B}(\vec{r},\omega)$ in the radiation zone.

4. Determine the power emitted per unit solid angle by the loop in the radiation zone.

2.7 Covariant Formalism of Electrodynamics (L11, L12)

2.7.1 Getting Familiar with Four-Vectors

In the following exercise, we will learn some basic four-vector manipulations. The greek indices $\mu, \nu, \dots$ take values $0, 1, \dots, d$, where d is the dimension of space:

1. Derive the position vector: Let now $x^\mu = \left(x^0, x^1, \dots, x^d\right)$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$.What is $\partial_\mu x^\mu$? Can you see that it is indeed a (Lorentz) scalar?

2. We can define a general tensor as an object with multiple indices, both up and down, i.e. $A^{\mu\nu\rho}_{\gamma\delta\sigma}$. Its transformation properties follow from those ones of the tensor product of vectors, i.e. $x'^\mu y'^\nu = \Lambda^\mu_\sigma \Lambda^\nu_\gamma x^\sigma y^\gamma$, which implies that $A'^{\mu\nu} = \Lambda^\mu_\sigma \Lambda^\nu_\gamma A^{\sigma\gamma}$.

   Prove however, that not every tensor can be written as a product of vectors. This means that it is not always possible to find a^μ, b^ν such that $\Sigma^{\mu\nu} a^\mu b^\nu$ (even if $S^{\mu\nu}$ is symmetric).

3. In order to distiguish between different tensors, we can tag them depending on their properties. In the following, let $A^{\mu\nu}$ be an antisymmetric tensor, that is $A^{\mu\nu} = -A^{\nu\mu}$ and $S^{\mu\nu}$ to be a symmetric tensor, so $S^{\mu\nu} = S^{\nu\mu}$.

   (a) Show that the (anti)symmetry property of a tensor is preserved by the Lorentz transformations.

   (b) Prove that $S^{\mu\nu} A_{\mu\nu} = 0$.

   (c) Let us now introduce the concept of symmetrization and antisymmetrization of a tensor with two indices. For an arbitrary tensor $C^{\mu\nu}$ we can define that $C^{(\mu\nu)} = \frac{1}{2}(C^{\mu\nu} + C^{\nu\mu})$. In the same spirit, its antisymmetrisation goes as $C^{[\mu\nu]} = \frac{1}{2}(C^{\mu\nu} - C^{\nu\mu})$.

   Show that a general tensor with two indices can be uniquely decomposed into the symmetric and antisymmetric part $C^{\mu\nu} = C^{(\mu\nu)} + C^{[\mu\nu]}$.

2.7.2 Covariant Formalism of Electrodynamics

1. Given the electromagnetic field tensor $F^{\mu\nu}$ with components

$$F^{0i} = -E^i, \quad F^{ij} = -\epsilon^{ijk} B_k, \quad F^{\mu\nu} = -F^{\nu\mu} \tag{2.7.1}$$

where $\epsilon_{123} = 1$, compute in terms of $\vec{E}$ and $\vec{B}$ fields the following tensor objects:

- $-F^{\mu\nu} F_{\mu\nu}$
- $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

2. Show that the Maxwell equations,

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0, \tag{2.7.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2.7.3}$$

are equivalent to the Bianchi identity $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$.

3. Given the energy-momentum tensor,

$$T^{\mu\nu} = F^\mu_\rho F^{\rho\nu} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}, \tag{2.7.4}$$

compute the components of T^{ij} in terms of $\vec{E}$ and $\vec{B}$ fields.

4. Show that the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is invariant under Lorentz transformations.

2.7.3 Lorentz Transformations for the Electromagnetic Field

1. Prove the general Lorentz transformation of the electric and the magnetic field.
2. Argue what happens to the angle between the electric and the magnetic field under a general boost transformation.

2.7.4 Three Observers. "One Field"

For some event, observer A measures $\mathbf{E} = (\alpha, 0, 0)$ and $\mathbf{B} = (\alpha, 0, 2\alpha,)$ and observer B measures $\mathbf{E}' = \left(E'_x, \alpha, 0\right)$ and $\mathbf{B}' = \left(\alpha, B'_y, \alpha,\right)$. Observer C moves with velocity $v\hat{\mathbf{x}}$ with respect to observer B.

Find:

1. the fields $\mathbf{E}'$ and $\mathbf{B}'$ measured by observer B.

2. the fields $\mathbf{E}''$ and $\mathbf{B}''$ measured by observer C.

2.7.5 Transformation of Force

A cylindrical column of electrons has uniform charge density ρ_0 and radius a.

1. Find the force on an electron at a radius $r < a$.

2. A moving observer sees the column as a beam of electrons, each moving with uniform speed $\mathbf{v}$. What force does this observer report is felt by an electron in the beam at a radius $r < a$?

2.7.6 A Long Wire Moving Fast

An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density q_0 in the inertial frame K'. The frame K' move with a velocity $\vec{v}$ parallel to the direction of the wire with respect to the laboratory frame K.

1. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.

2. What are the charge and current densities associated with the wire in its rest frame? In the laboratory?

3. From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part 1.

2.7.7 Relativistic Ohm's law

In the rest frame of a conducting medium the current density satisfies Ohm's law, $\vec{J}' = \sigma \vec{E}'$ in the rest frame.

1. Taking into account the possibility of convection current as well as conduction current, show that the covariant generalization of Ohm's law is

$$J^\mu - \frac{1}{c^2}\left(U_\nu J^\nu\right) U^\mu = \frac{\sigma}{c} F^{\mu\nu} U_\nu, \tag{2.7.5}$$

where U^μ is the 4 -velocity in the medium.

2. Find the 3 -vector current in a frame where the medium has velocity $\vec{v} = c\vec{\beta}$ with respect to some initial frame.

3. If the medium is uncharged in its rest frame, what is the charge density and the expression of the current density in the above frame.

2.7.8 E: A Loooooong Cylinder and Several Frames

1. An infinitely long cylinder of radius R has a uniform charge density ρ_0 and is at rest in an inertial frame K_0. The frame K_0 moves with a speed $\vec{v}$ parallel to the direction of the cylinder with respect to the laboratory frame K_L.

   (a) Find the electric field $\vec{E}_0$ and the magnetic field $\vec{B}_0$ in the rest frame (inside and outside the cylinder).

   (b) Find the electric field $\vec{E}_L$ and the magnetic field $\vec{B}_L$ in the frame of the laboratory (again both inside and outside the cylinder). Also find the current density $\vec{J}_L$ and the charge density ρ_L in the laboratory.

   (c) Add a second cylinder of radius R parallel to the first. The second cylinder carries a charge density ρ_L and current density $-\vec{J}_L$ in the frame of the laboratory. Let the distance between the axes of the two cylinders in the laboratory be $d > 2R$. Find the electric and magnetic fields outside the cylinders in the rest frame of the first cylinder K_0.

   (d) When there is only one cylinder is there an inertial reference frame where the electric field vanishes? In the situation with the two cylinders is there an inertial reference frame where the magnetic field $\vec{B}$ vanishes? Motivate your answers.

2. Consider the energy momentum tensor $T^{\mu\nu}(x)$ of some theory invariant under transla- tions and Lorentz transformations. The energy momentum is conserved i.e. $\partial_\mu T^{\mu\nu} = 0$.

   (a) Using the energy momentum tensor we can build a new object

   $$cM^{\mu\nu\rho}(x) = x^\rho T^{\mu\nu}(x) - x^\nu T^{\mu\rho}(x). \tag{2.7.6}$$

   Find what condition does $T^{\mu\nu}$ need to satisfy so that $\partial_\mu M^{\mu\nu\rho} = 0$. (that is $M^{\mu\nu\rho}$ is conserved.)

   (b) (For a bonus point) As seen in class the conserved four-momentum is an integral over space at any fixed time $P^\mu = \int_{t=\text{const}} d^3x T^{0\mu}$. Can you give an interpretation to the conserved quantities $N^{\nu\rho} = \int_{t=\text{const}} d^3xM^{0\mu\nu}$? Explain.

2.7.9 E: Planes and Frames

In an inertial frame K_0 there are two planes at $x_3 = 0$ and $x_3 = a$. The plane at $x_3 = 0$ carries a uniform charge surface density σ while the plane at $x_3 = a$ carries a uniform charge surface density $-\sigma$. Both planes are at rest in K_0. The frame K_0 moves with a speed $\vec{v} = v\hat{x}_1$ parallel to the x_1 axis with respect to the laboratory frame K_L.

1. Consider the electric field $\vec{E}_0$ in the inertial frame K_0. Assume that $\vec{E}_0$ vanishes for $x_3 < 0$. What is $\vec{E}_0$ between the two planes (that is for $0 < x_3 < a$) and in the region$x_3 > a$?.

2. Find the electric $\vec{E}_L$ and magnetic $\vec{B}_L$ fields in the frame of the laboratory K_L.

3. Find the charge surface densities on the two planes in the laboratory frame K_L.

4. Find the surface current densities on the two planes in the laboratory frame K_L.

5. Is there an inertial reference frame where the electric field $\vec{E}$ vanishes everywhere?

6. Consider the energy momentum tensor $T^{\mu\nu}(x)$ of some theory invariant under translations and Lorentz transformations. The energy momentum is conserved i.e. $\partial_\mu T^{\mu\nu} = 0$.

   (a) Using the energy momentum tensor we can build a new object

   $$D^\mu(x) = x_\nu T^{\mu\nu}(x). \qquad (2.7.7)$$

   Find what condition does $T^{\mu\nu}$ need to satisfy so that $\partial_\mu D^\mu = 0$. (that is D^μ is conserved.)

   (b) Is the condition you found satisfied by the energy momentum tensor of the electromagnetic fields $T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\rho}F^\nu_\rho + \frac{1}{4}g^{\mu\nu}F^{\rho\lambda}F_{\rho\lambda}\right)$?

2.7.10 E: Different Points of View

In an inertial reference frame there is an infinite long wire along the $\hat{z}$ direction. The wire is at rest and carries a nonzero linear charge density λ and a nonzero current $\vec{I} = I\hat{z}$.

1. Boost to a different inertial reference frame moving with speed $\vec{v} = v\hat{z}$ with respect to the rest frame of the wire. What is the linear charge density carried by the wire in the new reference frame? What is the current?

2. Under which condition on the values of λ and $\vec{I}$ in the rest frame of the wire is it possible to boost to a frame where the electric field produced by the wire vanishes? Similarly under which condition on the values of λ and $\vec{I}$ in the rest frame of the wire is it possible to boost to a frame where the magnetic field produced by the wire vanishes?

2.7.11 E: Waves Across Reference Frames

In an inertial reference frame K the electric and magnetic fields of an electromagnetic wave are given by

$$\vec{E} = \hat{z}Ce^{i(k_x x + k_y y - \omega t)}, \quad \vec{B} = \frac{c}{\omega}\left(k_y\hat{x} - k_x\hat{y}\right)Ce^{i(k_x x + k_y y - \omega t)}. \tag{2.7.8}$$

A second reference frame K' moves with speed $\vec{v} = v\hat{x}$ with respect to K. Let the origin of K and K' coincide at $t = t' = 0$.

1. Determine the electric and magnetic fields in the reference frame K' that is $\vec{E}'\left(x', y', z', t'\right)$ and $\vec{B}'\left(x', y', z', t'\right)$.

2. What is the direction of propagation of the wave in K'? what is its frequency?

2.8 Lagrangian Manipulation (L12, L13)

2.8.1 A Relativistic Particle Coupled to a Scalar Field

The action for a relativistic point particle coupled by a strength g to a space-time-dependent Lorentz scalar field $\varphi(x)$ is

$$S = -mc^2\int ds - g\int ds\varphi(\mathbf{r}(s)). \tag{2.8.1}$$

Find the equation of motion for the particle. How does the force on the particle differ from the Coulomb force of an electric field?

2.8.2 One-Dimensional Massive Scalar Field

A one-dimensional field theory with scalar potential $\varphi(x, t)$ is characterized by the action

$$S = \frac{1}{2}\iint dt\,dx \left[\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - m^2\varphi^2\right]. \tag{2.8.2}$$

Find the equation of motion for $\varphi(x,t)$ by both Lagrangian and Hamiltonian methods.

2.8.3 Introduction to Lagrangian Manipulations

An alternative Lagrangian density for the electromagnetic field[6] is,

$$\mathscr{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c} J_\alpha A^\alpha. \tag{2.8.3}$$

1. Derive the Euler-Lagrange equations of motion. Are they the Maxwell equations? Under what assumptions?

2. Show explicitly, and with those previous assumptions, that this Lagrangian density differs from the usual one[7] by a four-divergence. Does this divergence affect the action or the equations of motion?

2.8.4 Coupling Extra Fields

An axionic field [8] $a(x)$ is coupled to a gauge field $A_\mu(\vec{x})$ with an associated field strength $F_{\mu\nu}$. The action describing this system goes as:

$$\begin{aligned}\mathscr{S}[a(\vec{x}), A_\mu(\vec{x})] = &-\frac{1}{2}\int d^4\vec{x}\partial_\mu a \partial^\mu a - \frac{1}{4}\int d^4\vec{x} F^{\mu\nu}F_{\mu\nu} \\ &- \frac{1}{f}\int d^4\vec{x}\left[aF_{\mu\nu} * F^{\mu\nu} - 2\partial_\mu\left(aA_\nu * F^{\mu\nu}\right)\right].\end{aligned} \tag{2.8.4}$$

Where $*F$ is dual to F and f is a constant.

1. Under what circumstances is this action Lorentz invariant?

2. Find the Equations of Motion.

[6] The one you have seen during lectures and/or books.

[7] $\mathscr{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{c}J_\alpha A^\alpha$.

[8] Can be thought as a scalar. We will see in the solutions that indeed it needs to behave as a pseudoscalar field.

3. Show that $\mathscr{S}$ is invariant under a displacement of the axionic field as $a(\vec{x}) \to a(\vec{x}) + \epsilon$.

4. Calculate the Noether current associated to the previous displacement invariance.

2.8.5 E: Ponderous Light

Consider the following action for the four-potential A^μ and a scalar field ϕ.

$$S = \int d^4x \left(\frac{1}{8\pi} \left(\partial^\mu \phi - m A^\mu\right) \left(\partial_\mu \phi - m A_\mu\right) - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^\mu A_\mu \right), \tag{2.8.5}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and J^μ is a conserved current that is $\partial_\mu J^\mu = 0$.

1. Show that the action is invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \alpha$ provided that the scalar ϕ also shifts as $\phi \to \phi + m\,\alpha$. Gauge fix by imposing $\phi = 0$. Rewrite the action in this gauge.

2. Using the gauge fixed action write the equations of motion for A^μ.

3. By contracting the equations of motion with ∂_μ obtain an equation for $\partial_\mu A^\mu$. Use this equation to simplify the equations of motion.

4. Find the form of a plane wave solution to the equations of motion with no sources ($J^\mu = 0$). Given a wave-vector $\vec{k}$ what is the frequency of the wave? How many independent polarizations are there?

5. In the electrostatic case we have $\vec{A} = 0$. Find the electrostatic potential $\Phi = A^0$ due to a single electric charge q at rest at the origin. (Hint: you may try a solution of the form $\Phi(\vec{x}) = e^{-\alpha|\vec{x}|} f(\vec{x})$ for some function f and an appropriately chosen constant α)

2.9 Radiation and Relativistic Dynamics (L14)

2.9.1 Emission Rates by Lorentz Transformation

An electron enters and exits a capacitor with parallel-plate separation d through two small holes. The electron velocity is given by $v\hat{z}$ and it is parallel to the capacitor electric field $\vec{E}$. The change in the electron velocity is small. Calculate the total energy $\Delta U'_{EM}$ and its linear momentum $\Delta P'_{EM}$ that was radiated by the electron in both rest and laboraty frames (ΔU_{EM} and ΔP_{EM} respectively).

2.9.2 A Merry Go Round of Radiating Particles

N identical, equally spaced[9] point particles, each with a charge q, move in a circle of radius a. All of them have the same constant speed v around the ring. Show that the Lienard-Wiechert electric field is *static* everywhere on the symmetry axis.

2.9.3 The Direction of the Velocity Field

Prove that the "velocity" part of the Lienard-Wiechert electric field points to the observer from the "anticipated position" of the moving point charge. The latter is the position the charge *would* have moved *if* it retained the velocity $\vec{v}_{ret}$ from $t = t_{ret}$ to the present time of observation.

2.9.4 Radiating 14.4 Jackson Problem

Using the Liénard - Wiechert electric field, discuss the time-averaged power radiated per unit solid angle by a charged particle (e^-) in a **non-relativistic** motion in the next two different cases:

1. Along the z axis with position given by $z(t) = acos(\omega t)$,
2. In a circle of radius R in the plane xy with constant angular frecuency w_0.

2.9.5 A Fast Particle in a Constant Electric Field

A relativistic point particle with charge q and mass m moves in response to a uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$. The initial energy, linear momentum, and velocity are $\mathscr{E}_0$, p_0, and $\mathbf{u}(0) = u_0\hat{\mathbf{y}}$. Find $\mathbf{r}(t)$ and show that eliminating t gives the particle trajectory

$$z = \frac{\mathscr{E}_0}{qE}\cosh\left(\frac{qEy}{cp_0}\right). \tag{2.9.1}$$

Check the non-relativistic limit.

2.9.6 A Ringy Radiating Problem

1. A small current loop moves with constant velocity $\mathbf{v}_0$ as viewed in the laboratory frame. Find the vector potential $\mathbf{A}(\mathbf{r})$ and the scalar potential $\varphi(\mathbf{r})$ in the lab frame. It may be convenient to introduce the vector $\mathbf{R} = \mathbf{r} - \mathbf{v}_0 t$

[9] Is coronavirus still around?

2. Take the limit $v_0 \ll c$ in your formulae and deduce that the moving loop possesses both a magnetic dipole moment and an electric dipole moment.

3 Solutions

3.1 Electrostatics

3.1.1 Conducting ball

We have a conducting ball which is placed in an homogenous electric field $\vec{E} = E_z \hat{z}$. The starting point is to consider the general spherical solution for a "hollow" sphere on a constant field. Let's discuss this in depth.

This sphere has charge that is homogeneously distributed. On top of that, we can infer the following: If the field $\vec{E}$ is constant, the potential of this field should be linear with respect to z, as $-\vec{\nabla}\Phi = \vec{E}$. So $V \sim E_0 z\,\hat{z}$. But this would only be if the sphere was not present there. We also have to account for the field generated by the charge of the sphere and, hence, the potential of it. But here comes the trick. The potential generated by the sphere in negligible from far away. Why? We know that $\vec{E} = 0$ inside conductors, so the potential Φ inside of the sphere is a constant. By Gauss, we also know that the electric field generated by the sphere outside decreases proportionally as $\vec{E} \propto \frac{Q}{r^2}$. When $r \gg R$, with r being the observation position, we have that $\vec{E} \sim 0$.

So we know how the potential Φ (hence $\vec{E}$) looks like at two specific regimes. On the surface of the sphere this is:

$$\Phi_{r=R} = -\frac{Q}{4\pi\epsilon_0 R}, \tag{3.1.1}$$

and far far away from it, which goes as:

$$\Phi_{r\gg R} = \Phi_{\text{field}} + \Phi_{\text{sphere}} = E_0 \underbrace{r\cos\theta}_{z} - \cancel{\frac{Q}{4\pi\epsilon_0 R}}. \tag{3.1.2}$$

How does $\vec{E}$ look like in some mid region? Again, if we know the potential, we know the field. We have seen what the most general solution for the Laplace equation with azimuthal symmetry is given by:

$$\Phi(r,\theta) = \sum_{\ell=0} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta). \tag{3.1.3}$$

Where $P_\ell(\cos\theta)$ are Legendre polynomials... If we expand the first few terms of previous expression we can see that:

$$\Phi(r,\theta) = \underbrace{\left(A_0 + \frac{B_0}{r}\right)}_{f_0(r)} \underbrace{P_0}_{1} + \underbrace{\left(A_1 r + \frac{B_1}{r^2}\right)}_{f_1(r)} \underbrace{P_1}_{\cos\theta} + \cdots \tag{3.1.4}$$

it looks quite similar to the given Ansatz in the statement of the problem! In fact, we can now use this expression to fix the values of the coefficients A_ℓ, B_ℓ with help of boundary conditions $\Phi_{r=R}$ and $\Phi_{r\gg R}$.

$$\begin{aligned}
\Phi_{r=R} &= -\frac{Q}{4\pi\epsilon_0 R} = \left(A_0 + \frac{B_0}{R}\right) + \left(A_1 r + \frac{B_1}{R^2}\right)\cos\theta = \\
&= \cdots \text{matching powers of } R \cdots = \\
&\to B_0 = -\frac{Q}{4\pi\epsilon_0}, \\
&\to A_0 + \left(A_1 r + \frac{B_1}{R^2}\right)\cos\theta = 0.
\end{aligned} \tag{3.1.5}$$

If we use the boundary condition at ∞, we obtain that:

$$\begin{aligned}
\Phi_{r\to\infty} &= -E_0 r\cos\theta = A_0 + \cancel{\frac{B_0}{r}} + \left(A_1 r + \cancel{\frac{B_1}{r^2}}\right)\cos\theta = \\
&= \cdots \text{matching powers of } r \cdots = \\
&\to A_1 = -E_0, \\
&\to A_0 = 0.
\end{aligned} \tag{3.1.6}$$

We only need to fix the value of B_1. With all previous values fixed, go back to equation **??** to see obtain:

$$B_1 = E_0 R^3. \tag{3.1.7}$$

Then, all values together yield the following result:

$$\Phi(r,\theta) = -\frac{Q}{4\pi\epsilon_0 r} - E_0 R\cos\theta\left(\frac{r}{R} - \frac{R^2}{r^2}\right), \tag{3.1.8}$$

which one can easily see that behave as expected when $r \to 0$ and $r = R$. The only remaining thing is to compute the value of the electric field $\vec{E}$ given this potential. As we are dealing with an azimuthal symmetry, the gradient should be used in spherical coordinates. Then:

$$\vec{E} = -\left(\frac{Q}{4\pi\epsilon_0 r} + E_0 R\cos\theta\left(\frac{1}{R} + \frac{2R^2}{r^3}\right)\right)\hat{r} + \sin\theta E_0 R\left(\frac{1}{R} - \frac{R^2}{r^3}\right)\hat{\theta}. \tag{3.1.9}$$

3.1.2 Conducting ball Again

Let start by noting that the method of images consist of creating a fake charge q_f such that we can reproduce the result of the original set up in an easier way. In this specific case, as we have the sphere earthed, we already know that the potential ϕ on its surface is equal to 0 ($\phi(R)$ =0). Several points to consider in the current set up:

1. The charge q_t (t for true) can be located **anywhere** inside the sphere. The important thing is that the sphere is earthed.

2. As we have axial symmetry, this problem can be reduced to a two dimensional problem. We can use polar coordinates.

A rough sketch of this set-up considering a mirror images can be found in fig(**??**).

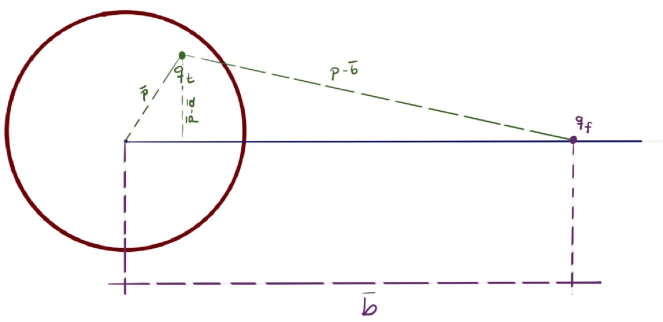


Figure 10: A rough sketch of this system studied by the method of images.

Then, the potential of this two charges, is given by superposition as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_t}{|\vec{r}_t|} + \frac{q_f}{\left|\vec{r}_f\right|}\right), \tag{3.1.10}$$

where q_f stands for false charge/mirror charge. Imposing the boundary condition on the surface $\phi = 0$, we have:

$$\frac{q_t}{\left|\vec{p} - \vec{a}\right|} = \frac{-q_f}{\left|\vec{p} - \vec{b}\right|}. \tag{3.1.11}$$

Now, we have to realise that we have two potential positions (a, b) to fix, but just one equations... This issue can be easily solved by accounting for two possible set ups:

1. Case where $p_x = R, p_y = 0$. In this case we will arrive to an equation that looks like:

$$\frac{q_t}{|R - a|} = \frac{-q_f}{\left|R - \vec{b}\right|} \rightarrow -\frac{q_t}{q_f} = \frac{R - a}{R - b}. \tag{3.1.12}$$

   Which is a good candidate as equation to solve for one of the variables. The other can be obtained by:

2. Case where $p_x = 0, p_y = R$.

   This will give another equation as:

$$-\frac{q_t}{q_f} = \sqrt{\frac{R^2 + a^2}{R^2 + b^2}}, \tag{3.1.13}$$

   which one can square both sides to get rid of the root.

From these two set-ups, one should realise that LHS of equations (**??**,**??**) are the same. Hence, equate them and manipulate the algebra to arrive to:

$$\left(R^2 + a^2\right) b - a\, b^2 = R^2 a. \tag{3.1.14}$$

This is a well-known second order equation that gives two solutions for b as: $b_1 = a$, which makes no sense, as it tells us to place both charges at the same position, and

$b_2 = \frac{R^2}{a}$, which relates a and b in a non-trivial way. Introduce this result into any of both previous cases (**??**, **??**) to find the relation between the charges expressed as:

$$q_f = -\frac{R\,q_t}{a} \tag{3.1.15}$$

With this result the potential ϕ follows as:

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_t}{\left|\vec{p}-\vec{a}\right|} - \frac{q_t\frac{R}{a}}{\left|\vec{p}-\left(\frac{R^2}{a},0\right)\right|}\right) = \tag{3.1.16}$$

$$= \frac{q_t}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{1}{\frac{a}{R}\sqrt{r^2+\frac{R^4}{a^2}-2r\frac{R^2}{a}\cos\theta}}\right). \tag{3.1.17}$$

The electric field $\vec{E}$ follows from taking the minus gradient of previous expression. To obtain the induced charge density, we just need to recall that it is given the derivative of the potential ϕ respect to the normal of the surface where we want to evaluate such density, i.e.

$$\sigma = -\epsilon\frac{\partial\phi}{\partial\vec{n}}, \tag{3.1.18}$$

In our case, due to spherical sym $\vec{n} = r$ and the surface of the sphere lies on $r = R$, so we just have to evaluate there. If one is careful with all the arithmetic and simplify cautiously, the result is:

$$\sigma = \frac{-q\epsilon}{4\pi\epsilon_0 aR}\left(\frac{1-\frac{R^2}{a^2}}{\sqrt[3]{\frac{R^2}{a^2}+1-2\frac{R}{a}\cos\theta}}\right). \tag{3.1.19}$$

Regarding the Gauss theorem for the electric field outside the sphere, it will the total charge inside the sphere divided by the total area where we evaluate the field.

Solutions b and c for this problems follows exactly the same steps as we have previously done. The main difference can be found in the boundary condition on the surface of the sphere. As this is not any longer connected to earth, the potential on the surface will be given by:

$$\phi(R) = V_0. \tag{3.1.20}$$

In any case, one can repeat both cases for different positions of the inner charge to extract that the new fake charge q'_f is:

$$q'_f = q_f + V_0. \tag{3.1.21}$$

From that point on, the rest of the exercise is straightforward to adapt to this new BC.

3.1.3 The Capacitance of an off-centered Capacitor

Given the description of the statement, the first thing should be to draw something similar to this sketch:

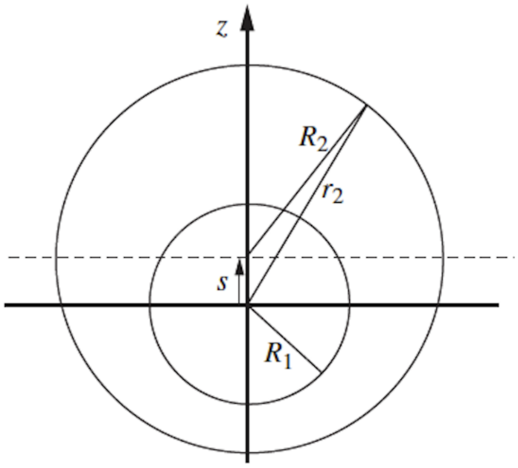


Figure 11: A rough sketch of the system we want to study....

1):

Let (r_2, θ) denote a point on the outer shell with respect to the origin of the inner shell (The small one). By the law of cosines, the difference between R_2^2 (big capacitor centre) and r^2 (small centre) is given by: $R_2^2 = r_2^2 + s^2 - 2r_2 s\cos\theta$. Therefore, expanding the square root of r_2 to first order in s, the boundary of the outer shell is:

$$r_2 = R_2 + s\cos\theta. \tag{3.1.22}$$

If the shells were exactly concentric, the potential between them would have the form $\varphi(r) = a + b/r$. Therefore, in light of the expansion to first order and the general solution of Laplace's equation in polar coordinates, we expect the potential in the space between the displaced shells to take the form[10]:

$$\varphi(r,\theta) = a + \frac{b}{r} + s\left(cr + \frac{d}{r^2}\right)\cos\theta + O\left(s^2\right) \tag{3.1.23}$$

To order s, fixing the boundary conditions at the shell surfaces we get

$$\begin{aligned} V_1 &= \varphi\left(R_1,\theta\right) = a + \frac{b}{R_1} + s\left(cR_1 + \frac{d}{R_1^2}\right)\cos\theta, \\ V_2 &= \varphi\left(r_2,\theta\right) = a + \frac{b}{R_2 + s\cos\theta} + s\left(c\left[R_2 + s\cos\theta\right] + \frac{d}{\left[R_2 + s\cos\theta\right]^2}\right)\cos\theta = \\ &= a + \frac{b}{R_2} + s\left(cR_2 + \frac{d}{R_2^2} - \frac{b}{R_2^2}\right)\cos\theta. \end{aligned} \tag{3.1.24}$$

We know that the potential on the BC V_1 and V_2 are constants, so the coefficients of $\cos\theta$ must vanish in (**??**). This fixes $d = -cR_1^3$ and $b = c\left(R_2^3 - R_1^3\right)$. Moreover, subtracting both conditions in (**??**) we get an extra equation as:

$$b = \left(V_1 - V_2\right) R_1 R_2 / \left(R_2 - R_1\right). \tag{3.1.25}$$

so c and d written in terms of R_i are:

$$c = \left(V_1 - V_2\right)\frac{R_1 R_2}{\left(R_2^3 - R_1^3\right)\left(R_2 - R_1\right)}, \qquad d = -\left(V_1 - V_2\right)\frac{R_1^4 R_2}{\left(R_2^3 - R_1^3\right)\left(R_2 - R_1\right)}. \tag{3.1.26}$$

Using (**??**), we can determine that the charge density on the surface of the inner shell is:

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R_1} = \epsilon_0 \frac{R_1 R_2\left(V_2 - V_1\right)}{R_2 - R_1}\left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3}\cos\theta\right]. \tag{3.1.27}$$

[10] This is an Ansatz of the Laplace equation. Observe that the first term is the zeroth order in the expansion in terms of Legendre polynomials for the most general solution, and the second term, is the subleading order, but, there is an "s" in front of everything. This is considered as a perturbation, as the inner sphere is slightly out of the centre.

The angular term in $\sigma(\theta)$ integrates to zero. Therefore, the total charge on the inner shell and the capacitance (to first order in s) are identical to the zero-order case of a concentric capacitor:

$$C_0 = \frac{Q}{V_1 - V_2} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1} \tag{3.1.28}$$

2):

By symmetry, there is only a z-component to the force on inner shell. Explicitly,

$$\mathbf{F} = \int dS \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int^{\pi} d\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0}\cos\theta = -\frac{Q^2}{4\pi\epsilon_0}\frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3} \tag{3.1.29}$$

3.1.4 Spherical cavity and spherical functions

1): First thing we should do in order to understand this problem, is to sketch how our geometrical distribution of potentials look like. The following sketch shows that:

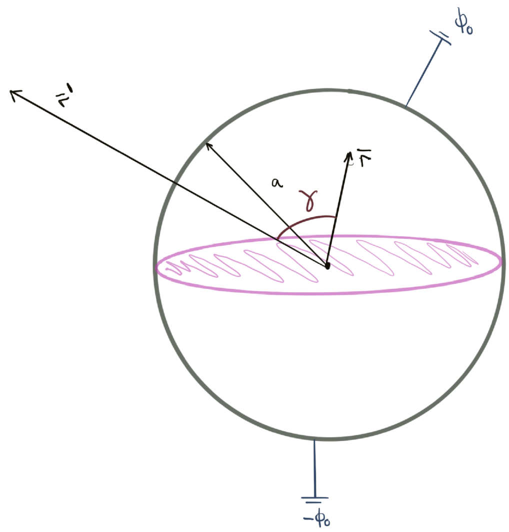


Figure 12: A rough sketch of the system we want to study....

We also know that the final appearance of the Green's function:

$$G\left(r,r'\right)=\underbrace{\frac{1}{\left|\vec{r}-\vec{r}'\right|}}_{G_1}-\underbrace{\frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}}_{G_2}. \tag{3.1.30}$$

So we have to basically massage two previous terms G_1 and G_2 to arrive to the desired result. Let's start by studying G_1. We know that, by addition theorem for spherical harmonics, G_1 can be expressed as:

$$G\left(r,r'\right)=\frac{1}{\left|\vec{r}-\vec{r}'\right|}=4\pi\sum_{l,m}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{l,m}^*\left(\theta',\phi'\right)Y_{l,m}(\theta,\phi). \tag{3.1.31}$$

So this part of the Green's function does not need more explanation. On the other hand, G_2 requires some changes before we can apply previous expression. Starting from:

$$G_2(r,r')=-\vec{r}'-\frac{a}{r'\left|\vec{r}-\frac{a^2}{r'^2}\vec{r}'\right|}, \tag{3.1.32}$$

We can expand the norm in the denominator, put r' inside the square root and extract an overall a from the square root to see:

$$G_2(r,r')=\frac{-\cancel{a}}{\sqrt{\cancel{a^2}\left(\frac{(r'r)^2}{a^2}+a^2-2r'r\cos\gamma\right)}}, \tag{3.1.33}$$

where γ is the angle between both vectors $\vec{r}$ and $\vec{r}'$. We can see rr' as a general vector and $\vec{a}$ as the vector position on the surface. If we undo square root to move back to an expression in terms of the norm, we immediately see that:

$$G_2(r,r')=\frac{-1}{\left|\frac{r\vec{r}'}{a}-\vec{a}\right|}. \tag{3.1.34}$$

From here, is easy to connect this expression with that of (**??**), as they have the same form. Introducing this new term in eq (**??**) we arrive to the conclusion that (massaging powers along the way):

$$G\left(r,r'\right)=4\pi\sum_{l,m}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y_{l,m}^*\left(\theta',\phi'\right)Y_{l,m}(\theta,\phi), \tag{3.1.35}$$

As we wanted to show.

2):

We would like to arrive to an expression of the form:

$$\Phi(r,\theta,\phi)=\sum_{lm}\frac{1}{a^2}\left(\frac{a}{r}\right)^{l+1}Y_{l,m}(\theta,\phi)\int\Phi_0\left(\theta',\phi'\right)Y_{l,m}^*\left(\theta',\phi'\right)d\Sigma', \tag{3.1.36}$$

This implies that we should start from the most general expression for a potential in terms of the Green's function, i.e:

$$\Phi(r,\theta,\phi)=\frac{1}{4\pi\epsilon_0}\int_V\rho(x')\,G(x,x')\,d^3x-\frac{1}{4\pi}\oint_S\Phi(x')\frac{\partial G}{\partial n'}dA'. \tag{3.1.37}$$

As we do not have any charge distribution in this given problem, this means $\rho(x')=0$. So the first term will not contribute. The next step, is to evaluate the interior of the remaining integral on the boundary where we can fix some parameters. In this case, the boundary condition is such that:

$$\left.\frac{\partial G}{\partial n'}\right|_{n'=a}\qquad\qquad\Phi(a,\theta,\phi)=\pm\phi_0. \tag{3.1.38}$$

Then, we have to evaluate the derivative of the Green's function G with respect to the normal $n'=r'$ on the surface of the sphere (i.e $r'=a$ after derivation). After some algebra and powers manipulation, the result is:

$$\left.\frac{\partial G}{\partial n'}\right|_{n'=a}=4\pi\sum_{l,m}\frac{l+1}{2l+1}\left(\frac{a^{l-1}}{r^{l+1}}\right)Y_{l,m}^*\left(\theta',\phi'\right)Y_{l,m}(\theta,\phi). \tag{3.1.39}$$

So taking this result and introducing inside expression (**??**), together with the boundary condition on the surface that $\Phi(a,\theta,\phi)=\pm\phi_0$, we obtained the final desired formula as:

$$\Phi(r,\theta,\phi) = \sum_{lm} \frac{l+1}{a^2(2l+1)} \left(\frac{a}{r}\right)^{l+1} Y_{l,m}(\theta,\phi) \int \Phi_0\left(\theta',\phi'\right) Y^*_{l,m}\left(\theta',\phi'\right) d\Sigma', \tag{3.1.40}$$

which obviously fades out when $r \to \infty$.

3.1.5 Green's function between concentric spheres

1):

First of all, we should proceed as always; To draw a sketch of the system we want to study:

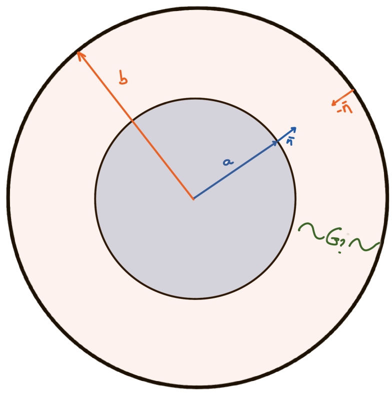


Figure 13: Some fancy sketch of the cavity we want to study.

This first part of the problem is asking us to show that, for a given concentric spherical geometry, the radial part of the Green's function looks like:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1} - a^{2l+1}} \left[\frac{l+1}{l} \left(r r'\right)^l + \frac{l}{l+1} \frac{(ab)^{2l+1}}{\left(r r'\right)^{l+1}} + a^{2l+1} \left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}} \right) \right]. \tag{3.1.41}$$

With a hint stating that any Green's function can be decomposed in its radial part and its spherical part, as a linear combination of spherical harmonics of the form:

$$G\left(x,x'\right)=\sum_{l=0}^{\infty}g_l\left(r,r'\right)P_l(\cos\gamma), \tag{3.1.42}$$

Furthermore, we know that the Neumann's boundary condition states that:

$$\frac{\partial}{\partial n'}G\left(x,x'\right)=-\frac{4\pi}{S}, \tag{3.1.43}$$

So we basically have all ingredients to solve this part of the problem. If we know the boundary condition appearance for the general Green's function, we also know it for its radial part. We just have to fix the coefficients A_l and B_l inside g_l making use of the double boundary condition. Double, because we have two surface where we can evaluate.

When evaluating on the inner sphere with radius a, we get:

$$\left.\frac{\partial G}{\partial n'}\right|_a=\sum_l\partial_r g_l(r,r')P_l(\cos\gamma)\Big|_a=\frac{1}{a^2+b^2}. \tag{3.1.44}$$

Where we have used that the normal is pointing outwards, so it has positive signature. It is also remarkable to realise the following; Our previous expression (**??**) has neither θ nor ϕ dependence, although there is a Legendre polynomial involved in its LHS. So this is already stating that the only set of spherical harmonics that will contribute in this problem to fix the value of the coefficients are those ones with $l=0$ (i.e $Y_{00}=\frac{1}{\sqrt{4\pi}}$.) This implies that one can express the radial part of the Green's function as:

$$\partial_{r'}g_l\Big|_{r'=a}=\frac{-1}{a^2+b^2}\delta_{l,0}, \tag{3.1.45}$$

Now, we can do exactly the same for the outer sphere. Here there is a crucial difference; The normal to this surface, as we want to evaluate the Green's function in the region between both spheres, is pointing *inwards*, so it will cary a negative sign. The same arguments we used for the inner one applies in this case too, so the result looks like:

$$\partial_{r'}g_l\Big|_{r'=b}=\frac{1}{a^2+b^2}\delta_{l,0}. \tag{3.1.46}$$

Equipped with this knowledge, let's the value of the coefficients. We know that:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + f_l\left(r,r'\right) = \frac{r_l^l}{r_>^{l+1}} + A_l r'^l + B_l r'^{-(l+1)}. \tag{3.1.47}$$

In this specific geometry we cannot drop any of the coefficients as we have done in previous exercises. This is due to the fact that we are now dealing with things happening inside and outside two different spheres that generate the given geometry. In any case, We have two boundary conditions values two fix two different equations, so we can solve for two different variables, A_l and B_l. Introducing evaluated green's radial function (**??**) and (**??**) as RHS of derivative with respect r of (**??**), we will get two expressions as:

$$\frac{l\,a^{l-1}}{r^{l+1}} + l\,a^{l-1}A_l + B_l\frac{-(l+1)\,a^l}{a^{2(l+1)}} = \frac{1}{a^2+b^2}\delta_{l,0}, \tag{3.1.48}$$

$$\frac{-(l+1)r^l}{b^{l+2}} + l\,b^{l-1}A_l + B_l\frac{-(l+1)\,b^l}{b^{2(l+1)}} = \frac{-1}{a^2+b^2}\delta_{l,0}. \tag{3.1.49}$$

Next step we have to perform, is just to solve for A_l and B_l in this coupled system of linear equations. Without loss of generality, we can set $l \neq 0$, so we get LHS of both expressions for the most general coefficients. The good part is that this simplifies RHS to 0. Solving then, one arrives to:

$$A_l = \frac{1}{a^{2l+1}+b^{2l+1}}\left(\frac{(l+1)\,r^l}{l} - \frac{a^{2l+1}}{r^{l+1}}\right), \tag{3.1.50}$$

$$B_l = \frac{-1}{b^{2l+1}-a^{2l+1}}\left(\frac{l\,(ab)^{2l+1}}{(l+1)r^{l+1}} + r^l a^{2l+1}\right). \tag{3.1.51}$$

The last step, after all despair and suffer we have gone through, it is just to introduce these values inside expression (**??**), massaging it a little bit to obtain:

$$g_l\left(r,r'\right) = \frac{r_<^l}{r_>^{l+1}} + \frac{1}{b^{2l+1}-a^{2l+1}}\left[\frac{l+1}{l}\left(r\,r'\right)^l + \frac{l}{l+1}\frac{(ab)^{2l+1}}{(r\,r')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right], \tag{3.1.52}$$

As we wanted to show.

2):

From our previous result, we can then continue in order to obtain a close expression for the potential $\Phi(r,\theta,\phi)$ in all the concentric region. Furthermore, with the potential, we can compute the value of the electric field $\vec{E}$ in that geometry. Hence, our starting point will be the expression given Neumann boundary conditions is:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\int_V \rho(x')\, G(x,x')\, d^3x + \frac{1}{4\pi}\oint_S G(r,r')\frac{\partial\Phi}{\partial n'}dA'. \tag{3.1.53}$$

As always, the first question we have to raise when this expression pops up is: *Is there any charge in our system?* As we do not have, this implies that $\rho = 0$, so the first term will not contribute and it is the second one that does. As in the previous part, the normal n' is given by the radius r'. Then, what is $\partial_{r'}\Phi$ inside previous expression? As we know, the gradient of the potential is minus the electric field $\vec{E}$, so in this case, we get that the derivative of Φ respect to $-r'$ is just the radial component of the electric field $\vec{E}$.

As we want to find a close expression for the potential, let us evaluate our previous expression for $r' = b$ (You can also do it at $r' = a$, but you will get no information, as $E_{r=a} = 0$). Recall the sign of the norm when computing the derivative.

$$\begin{aligned}\Phi(r' = b,\theta,\phi) &= \frac{1}{4\pi}\oint_S G(r,b) \underbrace{\frac{\partial\Phi}{\partial n'}}_{-E_r=E_0\cos\theta'}\left(\underbrace{b^2\sin\theta'}_{\text{jacobian}}\right)d\Omega', \\ &= \frac{1}{4\pi}E_0 b^2\oint G(r,b)\sin\theta'\cos\theta' d\Omega'.\end{aligned} \tag{3.1.54}$$

Inside of previous equation, we see our Green's function evaluated at b. Recall that this function can be decomposed in its radial part, as we shown in previous section (see eq(**??**)) and spherical harmonics. Introducing expression (**??**) in previous formula, and using eq(**??**) we obtain:

$$\Phi(b,\theta,\phi) = \frac{E_0 b^2}{4\pi}\sum_l \frac{4\pi}{2l+1}\, g_l(r,b)\oint_S \sum_m Y_{l,m}(\Omega)\, Y^*_{l',m'}(\Omega')\cos\theta'\sin\theta' d\Omega'. \tag{3.1.55}$$

Now we have a crucial point. A happy idea. Maria virgin, visiting us[11], to give us a hint. We have a cos in the game. We can exploit its presence and transform it into a spherical harmonic that can help us fix the previous expression. As you may know:

[11] This a rough translation of a say we have in Spanish.

$$Y_{1,0}(\Omega') = \sqrt{\frac{3}{4\pi}}\cos\theta'. \tag{3.1.56}$$

Introducing this in expression (**??**), and making use of spherical harmonics orthonormality, we obtain:

$$\Phi(b,\theta,\phi) = \sum_l \frac{E_0 b^2}{2l+1}\, g_l(r,b)\sqrt{\frac{4\pi}{3}}\, Y_{l,m}(\Omega)\,\delta_{m,0}\delta_{l,1}. \tag{3.1.57}$$

Simplifying annoying factors and evaluating $g_l(r,b)$ for $l = 1$, we arrive to the well deserved expression as:

$$\Phi(r,\theta) = E_0 \frac{r\cos\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right), \tag{3.1.58}$$

with notation $p = \frac{a}{b}$. We are just there. Now take this potential and derive it respect to θ and ϕ to compute those electric field components as:

$$E_r(r,\theta) = -E_0\frac{\cos\theta}{1-p^3}\left(1+\frac{a^3}{r^3}\right), \quad E_\theta(r,\theta) = E_0\frac{\sin\theta}{1-p^3}\left(1+\frac{a^3}{2r^3}\right), \quad E_\phi(r,\theta) = 0. \tag{3.1.59}$$

Finally, we are done. Congratulations to us. Take a rest, you deserve it. And some chocolate and/or fancy beverage. You deserve it even more!

3.2 Multipoles

3.2.1 Spherical Multiple Moment

1):

The first thing one should do is to sketch how the system looks like:

Given this charge distribution, we already know that the total charge of the system is $Q = 0$. Let's first prove that the charge distribution integrated over the whole space yields also a zero. We also know that the total charge is:

$$Q_T = \int \rho(\vec{x}')d^3x' = \int \rho(r',\theta',\phi')\, r'^2 \sin\theta' dr'\, d\phi'\, d\theta'. \tag{3.2.1}$$

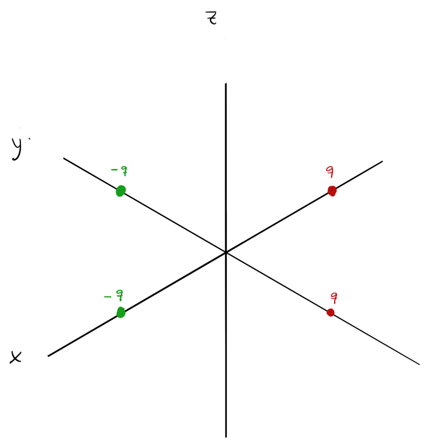


Figure 14: The distribution of the charges.

Recall that δ changes between Cartesian and spherical coordinates as:

$$\delta(\vec{x}' - \vec{x}) \to \frac{1}{r'^2 \sin\phi'}\delta(r' - r)\delta(\theta' - \theta)\delta(\phi' - \phi). \tag{3.2.2}$$

So the charge density for the four point charged particles is:

$$\begin{split}\rho(r') = \frac{q}{r'^2 \sin\phi'}(\delta(r' - a)\delta(\theta' - 0)\delta(\phi' - \tfrac{\pi}{2}) + \delta(r' - a)\delta(\theta' - \tfrac{\pi}{2})\delta(\phi' - \tfrac{\pi}{2}) - \\ \delta(r' - a)\delta(\theta' - \pi)\delta(\phi' - \tfrac{\pi}{2}) - \delta(r' - a)\delta(\theta' - \tfrac{3\pi}{2})\delta(\phi' - \tfrac{\pi}{2})).\end{split} \tag{3.2.3}$$

Which integrated over the whole space $r\epsilon[0, \infty], \theta\epsilon[0, 2\pi], \phi\epsilon[0, \pi]$ yields a total charge $Q_T = 0$ as expected. What about the quadrupoles?

2):

They also want us to calculate the multiple moments (recall: The charge density × the harmonics). This is given by the following formula as:

$$
\begin{aligned}
q_{lm} &= \int Y^*_{lm}(\theta',\phi') r'^l \rho(\vec{x}') d^3x' = \\
&= \int Y^*_{lm} r'^l \frac{q\,\delta(r'-a)\delta(\phi'-\frac{\pi}{2})}{r'^2 \sin\phi} \left(\delta(\theta'-0)+\delta(\theta'-\tfrac{\pi}{2})-\delta(\theta'-\pi)-\delta(\theta'-\tfrac{3\pi}{2})\right) r'^2 \sin\phi\, d^3x' = \\
&= q\,a^l \left(Y^*_{lm}(0,\tfrac{\pi}{2})+Y^*_{lm}(\tfrac{\pi}{2},\tfrac{\pi}{2})-Y^*_{lm}(\pi,\tfrac{\pi}{2})-Y^*_{lm}(\tfrac{3\pi}{2},\tfrac{\pi}{2})\right).
\end{aligned}
\tag{3.2.4}
$$

This can be further simplified, simply using the definition of spherical harmonics given by:

$$
Y_{lm}(\theta,\phi) = \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}} P_l^m(\cos\phi) e^{im\theta}, \qquad P_l^m(x) = (-1)^m(1-x^2)^{m/2} \tfrac{d^m}{dx^m} P_l(x).
\tag{3.2.5}
$$

So a more specific expression for q_{lm} is:

$$
q_{lm} = q\,a^l \sqrt{\tfrac{2l+1}{4\pi}\tfrac{(l-m)!}{(l+m)!}} P_l^m(\cos\tfrac{\pi}{2}) \left(1+(-i)^m-(-1)^m-i^m\right)
\tag{3.2.6}
$$

From here, one just have to compute the first non-zero entries. It follows that:

$$
q_{0,0} = q_{1,0} = q_{0,1} = 0, \qquad q_{1,1} = a\sqrt{\tfrac{3}{2\pi}}(1-i), \quad q_{1,-1} = -a\sqrt{\tfrac{3}{2\pi}}(1+i).
\tag{3.2.7}
$$

3.2.2 Multiple Moments in Cartesian Coordinates

UNDER CONSTRUCTION

3.2.3 Exterior Multipoles for a Specified Potential on a Sphere

1):

The general form of a spherical multipole expansion is given by:

$$
\Phi(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\Omega).
\tag{3.2.8}
$$

As we are asked to show the general form of an **exterior** multipole expansion, we have to get rid of $A_{\ell m}$ as this coefficient is only for $r < R$ cases. For $B_{\ell m}$ we have the basic description:

$$B_{\ell m} = \frac{4\pi}{2\ell+1} \int \rho\left(\mathrm{r}'\right) r'^{\ell}\, Y^{\star}_{\ell m}\left(\Omega'\right) dV'. \tag{3.2.9}$$

As we have to express eq(**??**) without the B coefficient, we need to find its explicit value for $\forall r > R$. Here we can abuse from spherical harmonic properties as:

$$\begin{aligned}
\Phi(R,\Omega) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} \frac{Y_{\ell m}(\Omega)}{R^{\ell+1}} = \\
&= \text{Multiply both sides times } Y^{\star}_{\ell' m'} \rightarrow \\
\int \Phi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega' &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{B_{\ell m}}{R^{\ell+1}} \int Y_{\ell m}\left(\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega'.
\end{aligned} \tag{3.2.10}$$

The orthonormality of the spherical harmonics gives the expansion coefficients as:

$$\begin{aligned}
&\int d\Omega\, Y^{\star}_{\ell',m'} Y_{\ell m} = \delta_{\ell'\ell}\,\delta_{m'm} \rightarrow \\
&\qquad B_{\ell m} = R^{\ell+1} \int \Phi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega'.
\end{aligned} \tag{3.2.11}$$

Then, all together looks like:

$$\Phi(\mathrm{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{R}{r}\right)^{\ell+1} Y_{\ell m}(\Omega) \int \varphi\left(R,\Omega'\right) Y^{\star}_{\ell' m'}\left(\Omega'\right) d\Omega', \quad r > R. \tag{3.2.12}$$

2):

We need to solve how the potential Φ looks like in asymptotic limit ($r \rightarrow \infty$) for the given distribution. In order to craft this, we have to find a linear combination of harmonics that can reproduce the behaviour of the combination of the octants. This can be done by brute force, integrating each of the different 8 regions for given solid angle or to think a little bit to reduce the required computation.

We can see that V changes to $\pm$ every $\pi/2$ in ϕ angle. We know that the associated number in the harmonics to this angle is m. Checking the values of $Y_{\ell m}$, we see that the change will occur when $m = \pm 2$, because:

$$Y_{\ell\pm2} \propto e^{\pm 2i\phi} \to e^{\pm i\pi} = \pm 1. \tag{3.2.13}$$

So, if $|m| = 2$, $\ell \geq 2$. We find a minimum value for ℓ. What about 2? Then, the associated spherical harmonic value reads:

$$Y_{22} \propto \sin^2\theta e^{2i\phi}. \tag{3.2.14}$$

This cannot be,as we want the associated value of θ to vary as $\pm$ when moving around the octants. But if we check the following harmonic:

$$Y_{3\pm2} = \tfrac{1}{4\pi}\sqrt{\tfrac{105}{2\pi}}\sin^2\theta\cos\theta e^{\pm 2i\phi}. \tag{3.2.15}$$

Where cos will allow that variation. Then $\Phi = \mathrm{L.C}\left(Y_{3,\pm2}\right)$. We are close to be able to offer an expression for the potential when $r \to \infty$. Recall also that $\Phi|_{r=R} = \pm V$ and when $r \to \infty$, only the smallest allowed value of ℓ will contribute in the leading order of the expression. With all this, we can state that:

$$\Phi(\mathrm{r}) = V\left(\frac{R}{r}\right)^4 2\sqrt{\frac{2\pi}{105}}\left(Y_{32} + Y_{3-2}\right) = V\left(\frac{R}{r}\right)^4 \sin^2\theta\cos\theta\cos 2\phi, \quad r \to \infty. \tag{3.2.16}$$

3.2.4 Radiating Fidget Spinner

So we want to study this system and get its values of $\mathbf{p}, \mathbf{m}$ and Q_{ij}. Let the distance from the axis of rotation to the charges be R, as displayed in the following sketch:

The electric dipole moment, as we know, is given by:

$$\mathbf{p} = \int d^3r\, \mathrm{r}\, \rho(\mathbf{r}, t). \tag{3.2.17}$$

Hence, we need to write down the position of the charges in this spinning device. We know that each of them are at $2\pi/3$ angular distance, so:

$$\begin{aligned}
\mathbf{x}_1 &= \left(R\cos(\omega t + 0), R\sin(\omega t + 0), 0\right),\\
\mathbf{x}_2 &= \left(R\cos\left(\omega t + \tfrac{2\pi}{3}\right), R\sin\left(\omega t + \tfrac{2\pi}{3}\right), 0\right),\\
\mathbf{x}_3 &= \left(R\cos\left(\omega t + \tfrac{4\pi}{3}\right), R\sin\left(\omega t + \tfrac{4\pi}{3}\right), 0\right).
\end{aligned} \tag{3.2.18}$$

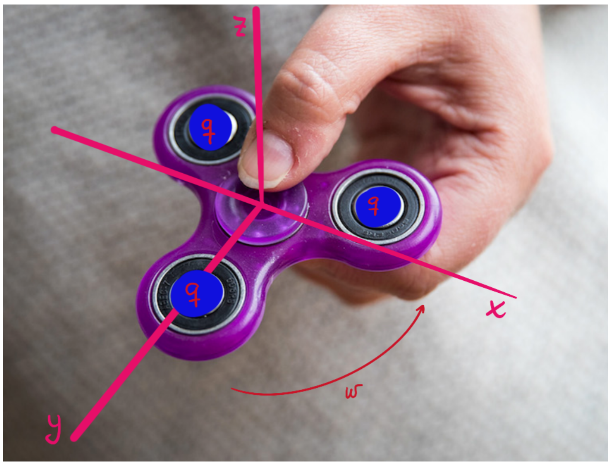


Figure 15: We can imagine we placed a charge in each of the inner holes.

We therefore have:

$$p_x = q\sum_{i=1}^{3} x_i = Rq(\cos(\omega t) + \cos(\omega t + 2\pi/3) + \cos(\omega t - 2\pi/3)) = 0 \tag{3.2.19}$$

and, similarly, $p_y = p_z = 0$. So, there is no electric dipole radiation.

The current density is given by $\mathbf{j}(\mathbf{r}, t) = \mathbf{v}\rho(\mathbf{r}, t)$ where the velocity $\mathbf{v}$ points (locally) in the direction of the particle motion with magnitude $v = R\omega$. The magnetic dipole moment can be calculated as:

$$\begin{aligned}\mathbf{m} = \int d^3r\, \mathbf{r}\times\mathbf{j} = \int d^3r \underbrace{\mathbf{r}\times\mathbf{v}}_{(x,y,0)\times(x,y,0)=\hat{z}} \rho(\mathbf{r}, t) = \\ = R^2\omega\,\hat{\mathbf{z}}\int d^3r\,\rho(\mathbf{r}, t) = 3\omega q R^2\,\hat{\mathbf{z}}.\end{aligned} \tag{3.2.20}$$

Observe that the magnetic dipole moment is time-independent! Therefore, there is no magnetic dipole radiation.

So far, no interesting properties for this system. What about the quadrupole momentum? The components of the electric quadrupole tensor are:

$$Q_{ij} = \int d^3x \left(3x^i x^j - r^2\delta^{ij}\right)\rho(\mathbf{x}, t). \tag{3.2.21}$$

The symmetry of ρ with respect to the x -axis dictates that $Q_{xy} = 0$ (Compute it yourself to check that everything nicely cancels). Since the charges all lie in the plane $z = 0$, we find that $Q_{xz} = Q_{yz} = Q_{zz} = 0$. So the only non-zero entries of Q_{ij} are:

$$\begin{aligned} Q_{xx} = q\sum_{i=1}^{3} x_i^2 = R^2 q\left(\cos^2(\omega t) + \cos^2(\omega t + 2\pi/3) + \cos^2(\omega t - 2\pi/3) - 1\right) = \\ = 3R^2 q(\cos(2\omega t) + \cos(2\omega t - 2\pi/3) + \cos(2\omega t + 2\pi/3)) = \frac{3}{2}R^2 q. \end{aligned} \tag{3.2.22}$$

and, recall that $Tr(Q) = 0$, so $Q_{yy} = -Q_{xx} = -\frac{3}{2}R^2 q$. Since Q is time-independent, there is no electric quadrupole radiation either.

3.3 Macroscopic Media

3.3.1 A Conducting Sphere at a Dielectric Boundary

We know that the general solution of Laplace's equation is given by:

$$\Phi(\mathbf{r},\theta,\phi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right) Y_{\ell m}(\Omega). \tag{3.3.1}$$

1):

Let the polar z -axis pass through the center of the sphere perpendicular to the dielectric interface. Then, the solution of Laplace's equation outside the sphere is:

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta). \tag{3.3.2}$$

At the sphere boundary, we must have $\Phi(R,\theta) = V = \text{const}$. This tells us that $B_\ell = 0$ for all $\ell \neq 0$ (as in the previous exercise, higher orders of r will not contribute) so:

$$\Phi(r,\theta) = \frac{B_0}{r} \Rightarrow \mathbf{E} = \frac{B_0}{r^2}\hat{\mathbf{r}}. \tag{3.3.3}$$

Therefore, wherever the dielectric constant is $\kappa_i (i = 1,2)$:

$$\mathbf{D}_i(r) = \epsilon_0 \kappa_i \frac{B_0}{r^2}\hat{\mathbf{r}}. \tag{3.3.4}$$

The constant B_0 can be obtained using one of Maxwell equations, $\nabla \cdot \mathbf{D} = \rho_c$. Using a spherical Gaussian surface,

$$\int_S d\mathbf{S} \cdot \mathbf{D} = \epsilon_0 B_0 2\pi \left[\kappa_1 \int_0^{\pi/2} d\theta \sin\theta + \kappa_2 \int_{\pi/2}^{\pi} d\theta \sin\theta \right] = 2\pi\epsilon_0 B_0 (\kappa_1 + \kappa_2) = Q. \quad (3.3.5)$$

Then we have:

$$\Phi(r) = \frac{Q}{2\pi\epsilon_0 (\kappa_1 + \kappa_2)} \frac{1}{r} \quad (3.3.6)$$

2):

The free charge on the surface of the sphere follows from Gauss' law as:

$$\sigma_c = \mathbf{D}(R) \cdot \hat{\mathbf{r}} = \begin{cases} \frac{\kappa_1}{\kappa_1+\kappa_2} \frac{Q}{2\pi R^2} & \text{in region } \kappa_1 \\ \frac{\kappa_2}{\kappa_1+\kappa_2} \frac{Q}{2\pi R^2} & \text{in region } \kappa_2 \end{cases} \quad (3.3.7)$$

There is polarization charge at the sphere boundary, as we have such a free charge distribution on the surface. Its value is $\sigma_P = (1-\kappa)\sigma_c/\kappa$. This charge is compensated by polarization charge at infinity. There is no polarization charge at the κ_1/κ_2 interface because $\mathbf{E}$ and hence $\mathbf{P}$ are everywhere radial. This means that $\mathbf{P} \cdot \hat{\mathbf{n}} = 0$ at the interface.

3.3.2 Polarization by Superposition

The Gauss' law electric field produced by a sphere with uniform charge density ρ centred at the origin is:

$$\mathbf{E}(r) = \begin{cases} \frac{\rho}{3\epsilon_0}\mathbf{r}, & r < R \\ \frac{\rho}{3\epsilon_0}\frac{R^3}{r^3}\mathbf{r} & r > R \end{cases} \quad (3.3.8)$$

An identical sphere, but with charge density $-\rho$ displaced from the origin by δ, produces the negative version of the previous field except that $\mathbf{r} \to \mathbf{r} - \delta$. With this in mind, the following can be approximated, such that:

$$
\begin{aligned}
|\mathbf{r}-\boldsymbol{\delta}|^{-3} = [(\mathbf{r}-\boldsymbol{\delta})\cdot(\mathbf{r}-\boldsymbol{\delta})]^{-3/2} = \\
&= \frac{1}{r^3}\left[1-\frac{2\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}+\frac{\delta^2}{r^2}\right]^{-3/2} = \\
&\approx \frac{1}{r^3}\left[1+\frac{3\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}\right].
\end{aligned}
\tag{3.3.9}
$$

Hence, the total field produced by the superposition of the two spheres is:

$$
\mathbf{E}(r) = \begin{cases} \frac{\rho}{3\epsilon_0}[\mathbf{r}-(\mathbf{r}-\boldsymbol{\delta})] = \frac{\rho\boldsymbol{\delta}}{3\epsilon_0} & r<R \\ \frac{\rho R^3}{3\epsilon_0}\left\{\frac{\mathbf{r}}{r^3}-\frac{\mathrm{r}-\delta}{r^3}\left[1+\frac{3\mathbf{r}\cdot\boldsymbol{\delta}}{r^2}\right]\right\} = \frac{\rho R^3}{3\epsilon_0}\left[\frac{\delta-3(\hat{\mathbf{r}}\cdot\boldsymbol{\delta})\hat{\mathbf{r}}}{r^3}\right] & r>R \end{cases}
\tag{3.3.10}
$$

Comparing these previous results with the field produced by a sphere with volume V and polarization $\mathbf{P}$:

$$
\mathbf{E}(r) = \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r<R \\ \frac{V}{4\pi\epsilon_0}\left[\frac{3(\hat{\mathbf{r}}\cdot\mathbf{P})\hat{\mathbf{r}}}{r^3}-\frac{\mathbf{P}}{r^3}\right] & r>R \end{cases}
\tag{3.3.11}
$$

We find that the two of them are identical if we identify $\mathbf{P} = -\rho\boldsymbol{\delta}$.

3.3.3 The Field at the Center of a Polarized Cube

Our starting point will be the field produced by a polarised object, which reads:

$$
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\int_V -\nabla'\mathbf{P}'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}dV' - \int_S dS'\mathbf{P}'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]
\tag{3.3.12}
$$

For this specific cubic case, we know that ρ_P is 0, as our cube is homogeneously polarised, so $\nabla\mathbf{P} = 0$. The surface polarization $\sigma_{\mathbf{P}} = \mathbf{P}\cdot\hat{\mathbf{n}}$ is P on the right (R) face of the cube and $-P$ on the left (L) face of the cube. Since we only have surface charge,

$$
\mathbf{E}(\mathbf{r}) = \frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \int_L dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right].
\tag{3.3.13}
$$

To simplify our lives, better to consider the value of all this when $\mathbf{r} = 0$. Then, at the origin:

$$
\mathbf{E}(0) = -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}'}{r'^3} - \int_L dS'\frac{\mathbf{r}'}{r'^3}\right].
\tag{3.3.14}
$$

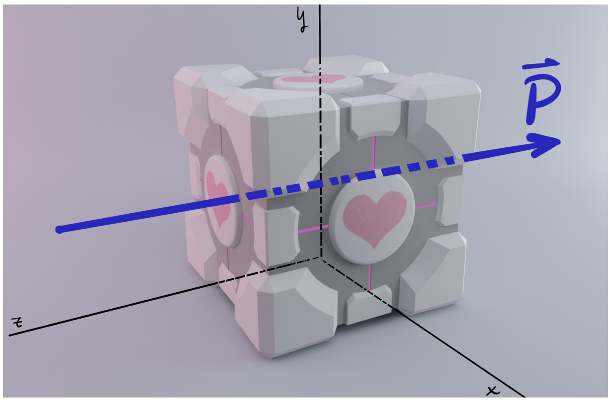


Figure 16: But the cake is still a lie...

By symmetry, the x and y components of these integrals are zero, as the polarisation only happens along the z-axis. Therefore, if the origin of the primed system is at the centre of the cube, we have:

$$\begin{aligned} E_z(0) &= -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{z'}{r'^3} - \int_L dS'\frac{z'}{r'^3}\right] = -\frac{2P}{4\pi\epsilon_0}\int_R dS'\frac{z'}{r'^3} = \\ &= -\frac{2P}{4\pi\epsilon_0}\int_R d\mathbf{S}'\cdot\frac{\mathbf{r}'}{r'^3} = \frac{2P}{4\pi\epsilon_0}\int_R d\Omega'. \end{aligned} \tag{3.3.15}$$

The integral is the solid angle subtended by the right face at the centre of the cube. By symmetry, this number must be $4\pi/6$. Therefore, the electric field at the centre of the cube is:

$$\mathrm{E}(0) = -\frac{\mathrm{P}}{3\epsilon_0} \tag{3.3.16}$$

This is exactly the same as the electric field at the centre of a uniformly polarized sphere.

3.3.4 E and D for an Annular Dielectric

1):

We are going to treat the geometry shown below as the superposition of a ball with radius b and uniform polarization P and a concentric ball with radius a and uniform polarization $-$P.

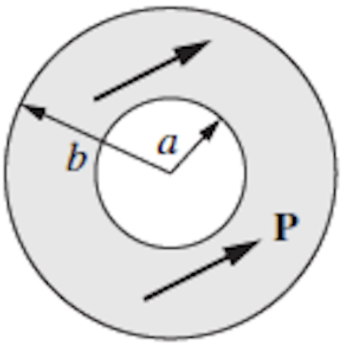


Figure 17: The two concentric spheres and the polarisation **P**.

From the text, the field produced by an origin-centered polarized ball with volume V is:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\frac{\mathbf{P}}{3\epsilon_0} & r < R \\ \frac{V}{4\pi\epsilon_0}\left\{\frac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & r > R \end{cases} \tag{3.3.17}$$

Therefore, the field we want to study is given by:

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0 & r < a, \\ -\frac{\mathbf{P}}{3\epsilon_0} - \frac{a^3}{3\epsilon_0}\left\{\frac{3(\mathrm{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & a < r < b, \\ \frac{b^3 - a^3}{3\epsilon_0}\left\{\frac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \frac{\mathbf{P}}{r^3}\right\} & r > b \end{cases} \tag{3.3.18}$$

2):

By symmetry, we should have $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$. Therefore, the choice of a spherical Gaussian surface of radius r gives:

$$\int_S d\mathbf{S}\cdot\mathbf{D} = D(r)4\pi r^2 = Q_{c,\text{encl}} = 0. \tag{3.3.19}$$

Therefore, $\mathbf{D} = 0$ everywhere.

3.3.5 E: A Charge and A Conducting Sphere

UNDER CONSTRUCTION

3.3.6 E: Critical strain

1):

The capacitor system with two plates can vary its width, as the dielectric filling the space between the plates is elastic. If we want to find an equilibrium point, we need to compute the minima of stability of the potential controlling this system. In this case we will have two different potential energies: The electric one, stored inside the capacitor itself and the mechanical one, given by the elastic properties of the "spring" between the plates. The electromagnetic energy stored inside a capacitor is given by:

$$U_{EM} = \frac{1}{2} A\, d \vec{E} \cdot \vec{D} = \frac{d\, q^2}{2\, A\epsilon}. \tag{3.3.20}$$

As the charge q is constant at equilibrium, the equilibrium position $d(q)$ will be given by just the derivative of the total energy U of the system derived respect to d, i.e.:

$$U_T = U_{EM} + U_{\text{elas}} \to \partial_d U_T = 0, \to d = d_0 - \frac{q^2}{2A\epsilon k}. \tag{3.3.21}$$

2):

The potential inside a capacitor is given the norm of the electric field times the separation of plates, as:

$$\Delta V = \|\vec{E}\| d = \frac{q\, d(q)}{A\epsilon} = \frac{q}{A\epsilon}\left(d_0 - \frac{q^2}{2A\epsilon k}\right). \tag{3.3.22}$$

Observe that this expression has two roots: $q = 0$ and $q = \sqrt{2A\epsilon k d_0}$. It also has a maximum for q lying at $q_{max} = \frac{\sqrt{3}\, q_0}{3}$. If we plot these results, we see that:

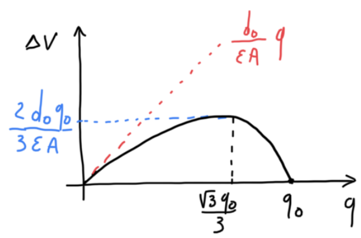


Figure 18: The potential for this elastic capacitor.

At the critical point q_0, the whole capacitor collapses, pointing to the instability of this system.

3.4 Light and Polarisation

3.4.1 Elliptic Polarisation Wave

The first good piece of news is that the magnetic part will not contribute, so less computation required. And we have the wave propagating in the z-direction.

1):

To show that $\vec{E}(0,t)$ parametrises an ellipse we just have to massage the given field components into:

$$\begin{aligned} E_x(0,t) &= A\cos(-\omega t), \\ E_y(0,t) &= B\cos(\phi - \omega t). \end{aligned} \tag{3.4.1}$$

We also know that an ellipse looks like:

$$a\,x^2 + 2\,b\,x\,y + c\,y^2 + f = 0, \qquad \forall\; a, b, c, f\,\epsilon \quad \mathbf{R}_+. \tag{3.4.2}$$

In this case, it is easy to think of both electric components as the x, y components of the ellipse. Let's square them to rewrite **??**.

$$\begin{aligned} E_x^2(0,t) &= A^2\cos^2(\omega t), \\ E_y(0,t) &= B^2\cos^2(\phi - \omega t). \end{aligned} \tag{3.4.3}$$

And now we prepare ourselves for a long boring trigonometrical computation[12]. We start from:

[12] Recall to use trigonometrical identities to simplify!

$$
\begin{aligned}
&a\,E_x^2 + 2\,b\,E_y\,E_x + c\,E_y^2 + f = 0,\\
&\left(\underbrace{a\,A^2 + 2\,b\,A\,B\,\cos(\phi) + c\,B^2\,\cos^2(\phi)}_{\hat{A}}\right)\underbrace{\cos^2(\omega\,t)}_{x^2} +\\
+&\left(\underbrace{c\,B^2\sin^2(\phi)}_{\hat{C}}\right)\underbrace{\sin^2(\omega\,t)}_{y^2} +\\
+&\left(\underbrace{2\,b\,A\,B\sin(\phi) + 2\,c\,B^2\cos(\phi)\sin(\phi)}_{\hat{B}}\right)\underbrace{\cos(\omega\,t)\sin(\omega\,t)}_{x\,y} + f = 0.
\end{aligned}
\tag{3.4.4}
$$

Where we have done all those identifications to mimic an ellipse equation. We are almost there. The main problem right now is that $\hat{A}, \hat{B}, \hat{C}$ have a dependence on ϕ. This means that they are not fixed and the appearance of the ellipse is not defined in the form of a circle. We have to fix those values.

How do we get a circle? In this case we have $\hat{B} = 0, \hat{A} = \hat{C}$ so the equation results into $x^2 + y^2 + \frac{f}{a} = 0$. The first and and most natural thought is to set $\phi = 0$, which makes $\hat{B} = 0$ but not $\hat{C}$... So we need something more involved, of the form:

$$
\begin{aligned}
\hat{B} = &\left(\underbrace{2\,b\,A\,B + 2\,c\,B^2\cos(\phi)}_{0}\right)\sin(\phi) \rightarrow\\
&\rightarrow\; b\,A\,B = -\,c\,B^2\cos(\phi).
\end{aligned}
\tag{3.4.5}
$$

We also need:

$$
\begin{aligned}
&\underbrace{a\,A^2 + 2\,b\,A\,B\,\cos(\phi) + c\,B^2\,\cos^2(\phi)}_{\hat{A}} = \underbrace{c\,B^2\sin^2(\phi)}_{\hat{C}},\\
&\rightarrow \quad \text{substitute } \textbf{??} \text{ and compute trigonometrics} \rightarrow\\
&a\,A^2 = c\,B^2.
\end{aligned}
\tag{3.4.6}
$$

Then, with **??** and **??** in mind, we go back to **??** to determine f as:

$$f = -a^2 A^2 \sin^2(\phi). \tag{3.4.7}$$

With all previous requirements we have a parametrisation of a circle as desired.

2):

Well, well, this part of the exercise looks like we have to change the basis of our system. The current description of the electric field is given by:

$$\mathbf{E}(\vec{x}, t) = \big(A\cos(kz-\omega t), B\cos(kz-\omega t+\phi), 0\big). \tag{3.4.8}$$

That we have to transform into something of the form:

$$\mathbf{E}_{\pm}(\vec{x}, t)_{New} = \Re\left[\mathbf{E}_{+}(z,t) + \mathbf{E}_{-}(z,t)\right]. \tag{3.4.9}$$

And we know how the electric field looks like in this new basis as:

$$\mathbf{E}_{\pm} = \tfrac{A_{\pm}}{\sqrt{2}}(\hat{x} \pm i\,\hat{y})e^{i(kz-\omega t)}. \tag{3.4.10}$$

So it seems that the only thing we have to do is to sum to prove that (**??**) is exactly the same as (**??**). Then:

$$\begin{aligned}
\Re\left[\mathbf{E}_{+} + \mathbf{E}_{-}\right] &= \Re\left[\left(\underbrace{\tfrac{A_{+}+A_{-}}{\sqrt{2}}}_{A} e^{i\cdots}\,\hat{x} + \underbrace{\tfrac{i(A_{+}-A_{-})}{\sqrt{2}}}_{B} e^{i\cdots}\,\hat{y}\right)\right] = \\
&= \Re\left[Ae^{i\cdots}\,\hat{x} + Be^{i\cdots}\,\hat{y}\right] = \\
&= \text{Expand with } e^{ix} = \cos x + i\sin x \text{ and let } \Re \text{ kill the imaginary terms to get} = \\
&= A\cos(kz-\omega t)\,\hat{x} - B\sin(kz-\omega t)\,\hat{y}.
\end{aligned} \tag{3.4.11}$$

If one reabsorbs that $-$ sign within the B one gets exactly the initial expression (**??**).

3.4.2 A Sandwich of Light

This problem contains the fundamental physics behind etalons, interferometers, and Fabry-Perot cavities[13]. There are three separate regions of uniform material, so we set up different electric fields in each and then relate them using boundary conditions. Because the incident wave is a plane wave, and the interfaces are flat, we assume the fields in all regions take on the form of plane waves (We do not want to complicate our lives). Let us call the incident material region "I", the air gap region "G", and the last slab region "F". Place one slice between the incident slab and the gap at $z = 0$ and the other interface at $z = d$. In the incident slab, there is a forward-going wave (the incident wave) and a backward-going wave (the sum of all reflected waves)[14].

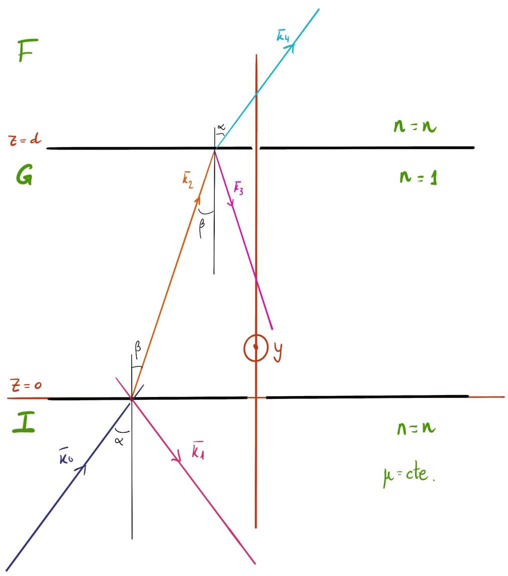


Figure 19: A plane wave trapped between two interphase.

In the gap there is also a forward-going wave (the sum of all forward-reflected waves) and a backward-going wave (the sum of all backward-reflected waves). In the transmitted slab there is only a forward-going wave (the sum of all transmitted waves). Note that all materials are lossless so that n and k are real-valued. The waves are all assumed to have linear polarization:

[13]Experimental, but important stuff. Not for the scope of this course.
[14]Observe the the picture.

$$
\begin{aligned}
\mathbf{E}_I &= \hat{\boldsymbol{\epsilon}}_0 E_0 e^{i\left(\frac{n}{c}\omega_0 \hat{\mathbf{k}}\cdot\mathbf{x}-\omega_0 t\right)} + \hat{\boldsymbol{\epsilon}}_1 E_1 e^{i\left(\frac{n}{c}\omega_1 \hat{\mathbf{k}}_1\cdot\mathbf{x}-\omega_1 t\right)},\\
\mathbf{E}_G &= \hat{\boldsymbol{\epsilon}}_2 E_2 e^{i\left(\frac{1}{c}\omega_2 \hat{\mathbf{k}}_2\cdot\mathbf{x}-\omega_2 t\right)} + \hat{\boldsymbol{\epsilon}}_3 E_3 e^{i\left(\frac{1}{c}\omega_3 \hat{\mathbf{k}}_3\cdot\mathbf{x}-\omega_3 t\right)},\\
\mathbf{E}_F &= \hat{\boldsymbol{\epsilon}}_4 E_4 e^{i\left(\frac{n}{c}\omega_4 \hat{\hat{k}}_4\cdot(\mathbf{x}-\mathbf{d})-\omega_4 t\right)}.
\end{aligned}
\tag{3.4.12}
$$

What are spatial boundary conditions? The boundary conditions must hold for all time and all points on the boundary. This means that the exponentials must match at $z = 0$ and $z = d$, leading to:

$$
\begin{aligned}
&\left[e^{i\left(\frac{n}{c}\omega_0 \hat{\mathbf{k}}\cdot\mathbf{x}-\omega_0 t\right)} = e^{i\left(\frac{n}{c}\omega_1 \hat{\mathbf{k}}_\mathbf{i}\cdot\mathbf{x}-\omega_1 t\right)} = e^{i\left(\frac{1}{c}\omega_2 \hat{\mathbf{k}}_2\cdot\mathbf{x}-\omega_2 t\right)} = e^{i\left(\frac{1}{c}\omega_3 \hat{\mathbf{k}}_\mathbf{3}\cdot\mathbf{x}-\omega_3 t\right)}\right]_{z=0},\\
&\left[e^{i\left(\frac{1}{c}\omega_2 \hat{\mathbf{k}}_2\cdot\mathbf{x}-\omega_2 t\right)} = e^{i\left(\frac{1}{c}\omega_3 \hat{\mathbf{k}}_3\cdot\mathbf{x}-\omega_3 t\right)} = e^{i\left(\frac{n}{c}\omega_4 \hat{\mathbf{k}}_4\cdot(\mathbf{x}-\mathbf{d})-\omega_4 t\right)}\right]_{z=d}
\end{aligned}
\tag{3.4.13}
$$

These two sets of equations must be true for all times t, so that the coefficients of t must match independently, leading to $\omega_0 = \omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega$. With the time components all cancelled out, we can simplify to something more handleable as:

$$
\left[n\hat{\mathbf{k}}\cdot\mathbf{x} = n\hat{\mathbf{k}}_1\cdot\mathbf{x} = \hat{\mathbf{k}}_\mathbf{2}\cdot\mathbf{x} = \hat{\mathbf{k}}_3\cdot\mathbf{x}\right]_{z=0} \text{ and } \left[\hat{\mathbf{k}}_\mathbf{2}\cdot\mathbf{x} = \hat{\mathbf{k}}_3\cdot\mathbf{x} = n\hat{\mathbf{k}}_4\cdot(\mathbf{x}-\mathbf{d})\right]_{z=d} \tag{3.4.14}
$$

All the wave vectors lie in the same plane. We can assume we have aligned the plane of incidence with the $x-z$ plane. As a result, none of the wave vectors have any y components. Expand the vectors into x and z components and define these components in terms of the angles from the z axis (for example $k_x = k\sin\theta_i, k_z = k\cos\theta_i$). Evaluate at $z = 0$ and $z = d$. Note that evaluating at specific z locations reduces the z-component equations down to just a bunch of constants. They have no meaning at this point because we can always suck a constant phase factor into the remaining undetermined coefficients E_0, E_1, etc. Because of the lack of meaningful information, we completely drop the z components. All that remains is the x components, which allow us to identify the angles as:

$$
\theta_I = \theta_R, \quad \theta_{G,I} = \theta_{G,R}, \quad \theta_I = \theta_{F,I}, \quad n\sin\theta_I = \sin\theta_{G,I}, \quad \sin\theta_{G,I} = n\sin\theta_{F,I}. \tag{3.4.15}
$$

Congratulations to us! We have just derived Snell's law. This will allow us to express all relations in terms of the incident angle. Note that because we are dealing with plane waves and flat interfaces, we can work with all these fields at the lateral position $x = 0$ without any loss of generality. In addition, we can evaluate the fields at time $t = 0$ without any loss of generality. Lastly, we use the shorthand notation $\cos\theta_{g,i} = \sqrt{1-n^2\sin^2\theta_i}$. With these simplifications, the fields become:

$$\begin{aligned}
\mathbf{E}_I &= \hat{\epsilon}_0 E_0 e^{i\frac{n}{c}\omega\cos\theta_i z} + \hat{\boldsymbol{\epsilon}}_1 E_1 e^{-i\frac{n}{c}\omega\cos\theta_i z},\\
\mathbf{E}_G &= \hat{\boldsymbol{\epsilon}}_2 E_2 e^{i\frac{1}{c}\omega\cos\theta_{g,i} z} + \hat{\boldsymbol{\epsilon}}_3 E_3 e^{-i\frac{1}{c}\omega\cos\theta_{g,i} z},\\
\mathbf{E}_F &= \hat{\epsilon}_4 E_4 e^{i\frac{n}{c}\omega\cos\theta_i (z-d)}.
\end{aligned} \tag{3.4.16}$$

This is going to be our starting point for the present waves in our system. We have two possible cases: The polarisation of the wave being $\perp$ to the plane of incidence and polarisation contained in the plane. Let's study both cases:

Perpendicular

This means that we leave the polarisation to fall in the $y-$direction. The electric and magnetic fields ($\mathbf{B} = \frac{n}{c}\hat{\mathbf{k}}\times\mathbf{E}$) are:

$$\begin{aligned}
\mathbf{E}_I &= \hat{\mathbf{y}} E_0 e^{i\frac{n}{c}\omega\cos\theta_i z} + \hat{\mathbf{y}} E_1 e^{-i\frac{n}{c}\omega\cos\theta_i z},\\
\mathbf{E}_G &= \hat{\mathbf{y}} E_2 e^{i\frac{1}{c}\omega\cos\theta_{g,i} z} + \hat{\mathbf{y}} E_3 e^{-i\frac{1}{c}\omega\cos\theta_{g,i} z},\\
\mathbf{E}_F &= \hat{\mathbf{y}} E_4 e^{i\frac{n}{c}\omega\cos\theta_i (z-d)},\\
\mathbf{B}_I &= \frac{n}{c}\left(\sin\theta_i\hat{\mathbf{z}} - \cos\theta_i\hat{\mathbf{x}}\right) E_0 e^{i\frac{n}{c}\omega\cos\theta_i z} + \frac{n}{c}\left(\sin\theta_i\hat{\mathbf{z}} + \cos\theta_i\hat{\mathbf{x}}\right) E_1 e^{-i\frac{n}{c}\omega\cos\theta_i z},\\
\mathbf{B}_G &= \frac{1}{c}\left(n\sin\theta_i\hat{\mathbf{z}} - \cos\theta_{g,i}\hat{\mathbf{x}}\right) E_2 e^{i\frac{1}{c}\omega\cos\theta_{g,i} z} + \frac{1}{c}\left(n\sin\theta_i\hat{\mathbf{z}} + \cos\theta_{g,i}\hat{\mathbf{x}}\right) E_3 e^{-i\frac{1}{c}\omega\cos\theta_{g,i} z},\\
\mathbf{B}_F &= \frac{n}{c}\left(\sin\theta_i\mathbf{z} - \cos\theta_i\mathbf{x}\right) E_4 e^{i\frac{n}{c}\omega\cos\theta_i (z-d)}.
\end{aligned} \tag{3.4.17}$$

Which will have to follow the boundary conditions[15] when no charges or currents are present:

$$\begin{aligned}
\left[\epsilon_2\mathbf{E}_2\cdot\mathbf{n} = \epsilon_1\mathbf{E}_1\cdot\mathbf{n}\right]_{z=0,d} \quad &, \quad \left[\mathbf{E}_2\times\mathbf{n} = \mathbf{E}_1\times\mathbf{n}\right]_{z=0,d},\\
\left[\mathbf{B}_2\cdot\mathbf{n} = \mathbf{B}_1\cdot\mathbf{n}\right]_{z=0,d} \quad &, \quad \left[\frac{1}{\mu_2}\mathbf{B}_2\times\mathbf{n} = \frac{1}{\mu_1}\mathbf{B}_1\times\mathbf{n}\right]_{z=0,d}.
\end{aligned} \tag{3.4.18}$$

[15] As we are dealing with dielectric materials, we must apply all boundary conditions.

So 8 boundary conditions in total. With good manners, patience and a beverage by your side, one can arrive to the following relations;

$$\begin{aligned}
&0 = 0 \quad \text{(2 equations will give this requirement)},\\
&E_2 + E_3 = E_0 + E_1 \quad \text{(2 equations will give this requirement)},\\
&E_2 - E_3 = b\,(E_0 - E_1)\,,\\
&E_4 = E_2 e^{ia} + E_3 e^{-ia} \quad \text{(2 equations will give this requirement)},\\
&bE_4 = E_2 e^{ia} - E_3 e^{-ia}.
\end{aligned} \tag{3.4.19}$$

where $b = \frac{n\cos\theta_i}{\sqrt{1-n^2\sin^2\theta_i}}$ and $a = \frac{1}{c}\omega d\sqrt{1-n^2\sin^2\theta_i}$.

Considering that the incident strength E_0 is taken to be a known, we have four independent equations above in four unknowns and can therefore solve uniquely for the different field strengths. After much algebra (mathematica in my case), we solve this system of equations as:

$$\begin{aligned}
\frac{E_1}{E_0} &= \frac{\left(1-b^2\right) i\sin a}{2b\cos a - \left(1+b^2\right) i\sin a},\\
\frac{E_2}{E_0} &= \frac{b(1+b)(\cos a - i\sin a)}{2b\cos a - i\left(1+b^2\right)\sin a},\\
\frac{E_3}{E_0} &= \frac{b(1-b)(\cos a + i\sin a)}{2b\cos a - i\left(1+b^2\right)\sin a},\\
\frac{E_4}{E_0} &= \frac{2b}{2b\cos a - i\left(1+b^2\right)\sin a}.
\end{aligned} \tag{3.4.20}$$

We now want to find the fraction of reflected and transmitted power by taking the magnitude squared of the first and last equation. We have to be careful because beyond the critical angle of total internal reflection, a and b become purely imaginary, but we can still have valid transmission via the evanescent modes. Let us approach the two cases separately. Below the critical angle, a and b are purely real-valued, leading to:

$$R = \left|\frac{E_1}{E_0}\right|^2 = \frac{\left(1-b^2\right)^2\sin^2 a}{4b^2\cos^2 a + \left(1+b^2\right)^2\sin^2 a}. \tag{3.4.21}$$

And the transmitted power is:

$$T = \left|\frac{E_4}{E_0}\right|^2 = \frac{4b^2}{4b^2\cos^2 a + \left(1+b^2\right)^2 \sin^2 a}. \tag{3.4.22}$$

As for the case we want to go beyond the critical angle of internal reflection, we have to notice than a, b become imaginary. We can rephrase them, for convenience, as $a = i\alpha$ and $b = -i\beta$ so the reflected and transmitted powers in this case become:

$$\begin{aligned} R &= \left|\frac{E_1}{E_0}\right|^2 = \frac{\left(1+\beta^2\right)^2 \sinh^2(\alpha)}{4\beta^2\cosh^2(\alpha) + \left(1-\beta^2\right)^2\sinh^2(\alpha)}, \\ T &= \left|\frac{E_4}{E_0}\right|^2 = \frac{4\beta^2}{4\beta^2\cosh^2(\alpha) + \left(1-\beta^2\right)^2\sinh^2(\alpha)}. \end{aligned} \tag{3.4.23}$$

Contained

We just have to do again the same thing for the polarization where the electric fields are in the plane of incidence. All of the forward going waves have **E** fields pointing in the negative- x/ positive- z direction and all the backwards going waves have **E** pointing in the positive- x/ positive- z direction. Using $\mathbf{B} = (n/c)\hat{\mathbf{k}} \times \mathbf{E}$, we find the fields for parallel polarization are:

$$\begin{aligned} \mathbf{E}_I &= (-\cos\theta_i\hat{\mathbf{x}} + \sin\theta_i\hat{\mathbf{z}})\, E_0 e^{i\frac{n}{c}\omega\cos\theta_i z} + (\cos\theta_i\hat{\mathbf{x}} + \sin\theta_i\hat{\mathbf{z}})\, E_1 e^{-i\frac{n}{c}\omega\cos\theta_i z}, \\ \mathbf{E}_G &= \left(-\cos\theta_{g,i}\hat{\mathbf{x}} + n\sin\theta_i\hat{\mathbf{z}}\right) E_2 e^{i\frac{1}{c}\omega\cos\theta_{g,i} z} + \left(\cos\theta_{g,i}\hat{\mathbf{x}} + n\sin\theta_i\hat{\mathbf{z}}\right) E_3 e^{-i\frac{1}{c}\omega\cos\theta_{g,i} z}, \\ \mathbf{E}_F &= (-\cos\theta_i\hat{\mathbf{x}} + \sin\theta_i\hat{\mathbf{z}})\, E_4 e^{i\frac{n}{c}\omega\cos\theta_i (z-d)}, \\ \mathbf{B}_I &= -\hat{\mathbf{y}}(n/c)\left(E_0 e^{i\frac{n}{c}\omega\cos\theta_i z} + E_1 e^{-i\frac{n}{c}\omega\cos\theta_i z}\right), \\ \mathbf{B}_G &= -\hat{\mathbf{y}}(1/c)\left(E_2 e^{i\frac{1}{c}\omega\cos\theta_{g,i} z} + E_3 e^{-i\frac{1}{c}\omega\cos\theta_{g,i} z}\right), \\ \mathbf{B}_F &= -\hat{\mathbf{y}}(n/c) E_4 e^{i\frac{n}{c}\omega\cos\theta_i (z-d)}. \end{aligned} \tag{3.4.24}$$

Where, again, $\cos\theta_{g,i} = \sqrt{1 - n^2\sin^2\theta_i}$. Imposing again eq(**??**), we will arrive to some relations of the amplitude of **E** and **B**. If one solves the system of four equations, the result is:

$$
\begin{aligned}
\frac{E_1}{E_0} &= \frac{\left(n^4 - b^2\right) i \sin a}{2n^2 b \cos a - i\left(n^4 + b^2\right)\sin a}, \\
\frac{E_2}{E_0} &= \frac{bn\left(n^2 + b\right)(\cos a - i \sin a)}{2n^2 b \cos a - i\left(n^4 + b^2\right)\sin a}, \\
\frac{E_3}{E_0} &= \frac{bn\left(n^2 - b\right)(\cos a + i \sin a)}{2n^2 b \cos a - i\left(n^4 + b^2\right)\sin a}, \\
\frac{E_4}{E_0} &= \frac{2n^2 b}{2n^2 b \cos a - i\left(n^4 + b^2\right)\sin a}.
\end{aligned}
\tag{3.4.25}
$$

Which again, being careful about the critical angle, we find below this:

$$
\begin{aligned}
R &= \left|\frac{E_1}{E_0}\right|^2 = \frac{\left(n^4 - b^2\right)^2 \sin^2 a}{4n^4 b^2 \cos^2 a + \left(n^4 + b^2\right)^2 \sin^2 a}, \\
T &= \left|\frac{E_4}{E_0}\right|^2 = \frac{4n^4 b^2}{4n^4 b^2 \cos^2 a + \left(n^4 + b^2\right)^2 \sin^2 a}.
\end{aligned}
\tag{3.4.26}
$$

With a, b as in the stated in the previous part of the exercise.

3.4.3 Faraday Rotation During Propagation

Let $k_L = \omega n_L / c$ and $k_R = \omega n_R / c$. Left and right circularly polarized plane waves with the same amplitude and frequency propagating along the z -axis in the medium we want to study. Then the plane waves should look like:

$$
\mathbf{E}_L(z,t) = E(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \exp\left[i\left(k_L z - \omega t\right)\right], \quad \mathbf{E}_R(z,t) = E(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \exp\left[i\left(k_R z - \omega t\right)\right]. \tag{3.4.27}
$$

In this basis, the given electric field is given by:

$$
\mathbf{E}(z=0,t) = \frac{1}{2}\left[\mathbf{E}_L(z=0,t) + \mathbf{E}_R(z=0,t)\right]. \tag{3.4.28}
$$

Therefore, at other points in space we find:

$$
\begin{aligned}
\mathbf{E}(z,t) &= \frac{1}{2}\left[\mathbf{E}_L(z,t)+\mathbf{E}_R(z,t)\right], \\
&= \frac{1}{2}E\left[e^{ik_Lz}+e^{ik_Rz}\right]e^{-i\omega t}\hat{\mathbf{x}}+\frac{i}{2}E\left[e^{ik_Lz}-e^{ik_Rz}\right]e^{-i\omega t}\hat{\mathbf{y}}.
\end{aligned}
\tag{3.4.29}
$$

The field will be linearly polarized along $\hat{\mathbf{y}}$ when:

$$
e^{ik_Lz} = -e^{ik_Rz} = e^{ik_Rz}e^{\pm im\pi}, \quad m = 1,3,5,\ldots \tag{3.4.30}
$$

And this only happens when z takes values of the form:

$$
z = \pm\frac{m\pi}{k_L-k_R} = \pm\frac{m\pi c/\omega}{n_L-n_R}, \quad m = 1,3,5,\ldots \tag{3.4.31}
$$

3.4.4 Charged Particle Motion in a Circular Polarized wave

a):

The physical electric field is:

$$
\mathrm{E}(z,t) = (\hat{\mathbf{x}}+i\hat{\mathbf{y}})E_0e^{+i(kz-\omega t)}+(\hat{\mathbf{x}}-i\hat{\mathbf{y}})E_0e^{-i(kz-\omega t)}. \tag{3.4.32}
$$

And the corresponding magnetic field can be obtained from:

$$
\mathrm{B}(z,t) = \frac{1}{c}\hat{\mathbf{z}}\times\mathrm{E}(z,t). \tag{3.4.33}
$$

Therefore (Long computation):

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{q}{m}\left[\mathbf{E}+\mathbf{v}\times\frac{1}{c}(\hat{\mathbf{z}}\times\mathbf{E})\right] = \\
&= \frac{q}{m}\left[\left(1-\frac{v_z}{c}\right)\mathbf{E}+\hat{\mathbf{z}}\frac{\mathbf{v}\cdot\mathbf{E}}{c}\right] = \\
&= \frac{q}{m}\left(1-\frac{v_z}{c}\right)E_0\left\{(\hat{\mathbf{x}}+i\hat{\mathbf{y}})E_0e^{+i(kz-\omega t)}+(\hat{\mathbf{x}}-i\hat{\mathbf{y}})E_0e^{-i(kz-\omega t)}\right\}+ \\
&\quad +\hat{\mathbf{z}}\frac{qE_0}{mc}\left\{\left(v_x+iv_y\right)e^{+i(kz-\omega t)}+\left(v_x-iv_y\right)e^{-i(kz-\omega t)}\right\}.
\end{aligned}
\tag{3.4.34}
$$

So the required velocities are:

$$
\begin{aligned}
\frac{dv_z}{dt} &= \frac{1}{2}\Omega\left\{v_+ e^{+i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right\}, \\
\frac{dv_\pm}{dt} &= \Omega\left(c - v_z\right) e^{\mp i(kz-\omega t)},
\end{aligned}
\tag{3.4.35}
$$

Where $v_\pm = v_x \pm i v_y$ and $\Omega = 2qE_0/mc$.

b):

Now define $\ell_\pm = v_\pm e^{\pm i(kz-\omega t)} \pm ic\Omega/\omega$ so

$$
\frac{dv_z}{dt} = \frac{1}{2}\Omega\left(\ell_+ + \ell_-\right) \tag{3.4.36}
$$

On the other hand,

$$
\frac{d\ell_\pm}{dt} = \frac{dv_\pm}{dt} e^{\pm i(kz-\omega t)} \mp i\omega v_\pm e^{\pm i(kz-\omega t)} = \Omega\left(c - v_z\right) \mp i\omega v_\pm e^{\pm i(kz-\omega t)}. \tag{3.4.37}
$$

Therefore, using (**??**),

$$
\frac{d}{dt}\left(\ell_- - \ell_+\right) = -i\omega\left[v_+ e^{i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right] = -\frac{2i\omega}{\Omega}\frac{dv_z}{dt}. \tag{3.4.38}
$$

So we can conclude that:

$$
\frac{d}{dt}\left\{v_z - i\frac{\Omega}{2\omega}\left(l_+ - \ell_-\right)\right\} = 0 \tag{3.4.39}
$$

Hence, a constant of the motion is:

$$
K = v_z(0) - i\frac{\Omega}{2\omega}\left[l_+(0) - \ell_-(0)\right]. \tag{3.4.40}
$$

c):

Differentiating (**??**) gives:

$$\frac{d^2 v_z}{dt^2} = \frac{\Omega}{2}\left[\frac{d\ell_+}{dt} + \frac{d\ell_-}{dt}\right] = \Omega^2 (c - v_z) - \frac{1}{2} i\omega\Omega \left\{ v_+ e^{i(kz-\omega t)} - v_- e^{-i(kz-\omega t)} \right\}. \quad (3.4.41)$$

But, recall that $\ell_+ - \ell_- = v_+ e^{i(kz-\omega t)} - v_- e^{-i(kz-\omega t)} + 2i\Omega c/\omega$ so

$$\frac{d^2 v_z}{dt^2} = \Omega^2 (c - v_z) - \frac{1}{2} i\omega\Omega \{\ell_+ - \ell_- - 2i\Omega c/\omega\} = -\left(\Omega^2 + \omega^2\right) v_z + \omega^2 K. \quad (3.4.42)$$

Now, imposing some initial conditions as $v(0) = 0$ and $\ell_\pm(0) = \pm i c\Omega/\omega$, in which case, $\omega^2 K = c\Omega^2$. Hence, if we define:

$$P = c\Omega^2 \quad \text{and} \quad \Omega_0^2 = \Omega^2 + \omega^2 \quad (3.4.43)$$

The EOM for v_z is

$$\frac{d^2 v_z}{dt^2} + \Omega_0^2 v_z = P. \quad (3.4.44)$$

This is solved by writing

$$\frac{d^2}{dt^2}\left(v_z - \frac{P}{\Omega_0^2}\right) + \Omega_0^2 \left(v_z - \frac{P}{\Omega_0^2}\right) = 0. \quad (3.4.45)$$

so $v_z(t) = A \sin\Omega_0 t + B\cos\Omega_0 t + P/\Omega_0^2$. The initial conditions $v_z(0) = \dot{v}_z(0) = 0$ determine the constants and we finally can get:

$$\begin{aligned} v_z(t) &= \frac{P}{\Omega_0^2}(1 - \cos\Omega_0 t), \\ a_z(t) &= \frac{P}{\Omega_0^2}\sin\Omega_0 t. \end{aligned} \quad (3.4.46)$$

No steady acceleration occurs; the particle cyclically accelerates and decelerates as it propagates along the z -axis.

3.4.5 E: A Wave and Some Boundary Conditions

1):

The first thing we should do, as always, is to draw a sketch of the geometrical set-up is described in the statement of the problem. It roughly looks like:

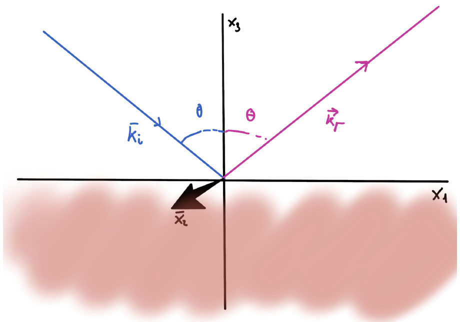


Figure 20: A sketch of the planes and waves in this problem.

With fields given by:

$$\vec{E}_i(\vec{x},t) = \vec{E}_0 e^{\mathrm{i}\vec{k}\cdot\vec{x}-\mathrm{i}\omega t}, \tag{3.4.47a}$$

$$\vec{B}_i(\vec{x},t) = \frac{\hat{k}}{c} \times \vec{E}_i, \tag{3.4.47b}$$

where $\{i,r\}$ sub-indices stand for incident and reflected waves. Let us now suppose that we have the incident wave on a perfect conductor at $x_3 = 0$. The plane of incidence is fixed by $\hat{k}$ and $\hat{x}_3$. As we did in the previous problem *A Sandwich of light*, we will have to consider perpendicular and parallel polarization with respect the plane that the wave vector and $\hat{x}_3$ form. Without any loss of generality, the wave vector can be written as:

$$\vec{k} = k_2\hat{x}_2 - k_3\hat{x}_3. \tag{3.4.48}$$

We can always rotate the system such that the wave vector is contained in the plane $\hat{x}_2 - \hat{x}_3$. The problem does not state any other type of material in $x_3 < 0$, so we keep the same permeabilities as in $x_3 > 0$ ($\mu_2 = \mu_1$ and $\epsilon_2 = \epsilon_1$). Now that we have $\vec{k}$ fixed in some plane, let us check the boundary conditions at $x_3 = 0$. That for the wave vector takes the well known form of:

$$[\mathbf{k}_i \cdot \mathbf{x} = \mathbf{k}_r \cdot \mathbf{x}]_{x_3=0} \tag{3.4.49}$$

While the boundary condition for the component of fields **E** and **B** are given at $x_3 = 0$ by:

$$\begin{aligned} &[\mathbf{E}_1 \cdot \mathbf{n} = \mathbf{E}_2 \cdot \mathbf{n}]_{x_3=0} \quad , \quad [\mathbf{E}_1 \times \mathbf{n} = \mathbf{E}_2 \times \mathbf{n}]_{x_3=0}\,, \\ &[\mathbf{B}_1 \cdot \mathbf{n} = \mathbf{B}_2 \cdot \mathbf{n}]_{x_3=0} \quad , \quad [\mathbf{B}_1 \times \mathbf{n} = \mathbf{B}_2 \times \mathbf{n}]_{x_3=0}\,. \end{aligned} \tag{3.4.50}$$

These both packages have to be satisfied at any moment in time[16]. We can notice that, although the norm form both wave vector $\mathbf{k}_{i,r}$ is the same, the x_3 will change in sign, as the wave is reflected. All in all, the most general expression for **E**, for this region of space that we can think of, is a linear combination of the incoming wave and the reflected one, as:

$$\mathbf{E}_1 = \vec{E}_{0i} e^{i(k_2\hat{x}_2 - k_3\hat{x}_3 - \omega t)} + \vec{E}_{0r} e^{i(k_2\hat{x}_2 + k_3\hat{x}_3 - \omega t)} \tag{3.4.51}$$

Observe that we have not specified yet the polarisation of this field. This depends if we want to consider it perpendicular o parallel to the incidence plane. Let's do that now:

PERPENDICULAR

In this case, that means that $\vec{E}_{0i,r} = E_{0i,r}\hat{x}_1$. Then, imposing boundary conditions (**??**) we obtain that:

$$\begin{aligned} \mathbf{E}_1 \times \hat{n} = 0 \qquad &\rightarrow \qquad E_{0i} = -E_{0r} \\ \mathbf{E}_{1\perp} = -2iE_{0i}\hat{x}_1 \sin&(k_3 x_3)\, e^{i(k_2 x_2 - \omega t)}, \end{aligned} \tag{3.4.52}$$

[16]But notice that we are dealing with a conducting material. This means that there should not be any fields **inside**, but there can be fields outside, so $E \cdot n$ does not make sense to set to 0. On top of that, as we have a perfect conducting material, any incident electric field on the surface can induce a current $\vec{j}$, with surface energy density σ. This can be observed checking the four boundary condition equations from Maxwell. Although to discuss this is not the scope of the problem, it is important to notice it.

where we set $\mathbf{E}_2 = 0$ as there is not refracted field below $x_3 = 0$. For the magnetic field $\mathbf{B}_1$ we can just take our previous result and compute:

$$
\begin{aligned}
\mathbf{B}_{i\,\perp} &= \frac{\vec{k}_i \times \mathbf{E}_{i\perp}}{c} = -\frac{k_3\hat{x}_2 + k_2\hat{x}_3}{c} E_{0i}\, e^{i(k_2\hat{x}_2 - k_3\hat{x}_3 - \omega t)}, \\
\mathbf{B}_{r\,\perp} &= \frac{\vec{k}_r \times \mathbf{E}_{r\perp}}{c} = -\frac{k_3\hat{x}_2 - k_2\hat{x}_3}{c} E_{0i}\, e^{i(k_2\hat{x}_2 + k_3\hat{x}_3 - \omega t)}.
\end{aligned} \tag{3.4.53}
$$

Which results in the total wave:

$$
\mathbf{B}_{1\,\perp} = -2\,c^{-1}\left(k_3\hat{x}_2 \cos(k_3x_3) - i\,k_2\hat{x}_3 \sin(k_3x_3)\right)\,E_{0i}\, e^{i(k_2\hat{x}_2 - \omega t)}. \tag{3.4.54}
$$

PARALLEL

In this case, we just have to write the the polarisation contained in the plane $\hat{x}_2 - \hat{x}_3$. This will depend on the orientation of the wave vector of each of the waves. We can just take our previous polarisation result and:

$$
\mathbf{k}_{i,r} \times \vec{\epsilon}_1 = \frac{-k_3\hat{x}_2 \mp k_2\hat{x}_3}{k}, \tag{3.4.55}
$$

where k stands for the norm of the wave vector. If we then repeat previous steps, we obtain:

$$
\begin{aligned}
\mathbf{E}_{1\,\parallel} &= \frac{2E_0}{k}\left(-\cos(k_3x_3)\,k_2\hat{x}_3 + i\sin(k_3x_3)\,k_3\hat{x}_2\right) e^{i(k_2x_2 - \omega t)}, \\
\mathbf{B}_{1\,\parallel} &= 2E_0\hat{x}_1 \cos(k_3x_3)\, e^{i(k_2x_2 - \omega t)}.
\end{aligned} \tag{3.4.56}
$$

2):

What if we now set an extra interphase at a distance d from the previous one? The first thing we will notice is that this problem starts looking more and more to *A Sandwich of Light*. Assuming there is no transmission to what may be above $x_3 = d$, let's compute the boundary conditions (**??**) for the fields. If we study now the **perpendicular** case, we will see that:

$$
[\mathbf{E}_{1\,\perp} \times \hat{n}]_{x_3=d} = 0 \qquad \longrightarrow \qquad k_3 = \frac{n\pi}{d}, \quad \forall\, n \in \mathbb{Z}. \tag{3.4.57}
$$

The same result applies to the **parallel** case. So the wavelength must be integer related to the height of the gap between both interphase.

3):

We can even further complicate our lives and consider that both conducting planes form a 90 degrees angle. We can then study waves contained in the quarter of place $x_3 > 0$ and $x_1 > 0$. The question we raise now is: How many waves, linearly combined, we need to described a system like this? If we were naive enough, we would consider a $1D$ wave (a line) hitting one of the two planes and bouncing twice on the planes, creating 3 arrays that combined, will give us an answer. Buuuuut, we are not so naive, are we? We know that plane waves are not straight lines, but two dimensional planes propagating in spacetime. Hence, a wave like this, will hit at the same time both conducting planes, and we will have two first bouncing lines: One propagating from vertical plane to the horizontal one and other one moving in the opposite direction. Both of them will hit the other plane at the same time, and bounce back in the opposite direction to the initial wave came from. In any case, it is better to see what this paragraph means than to interpret. Observe the following sketch:

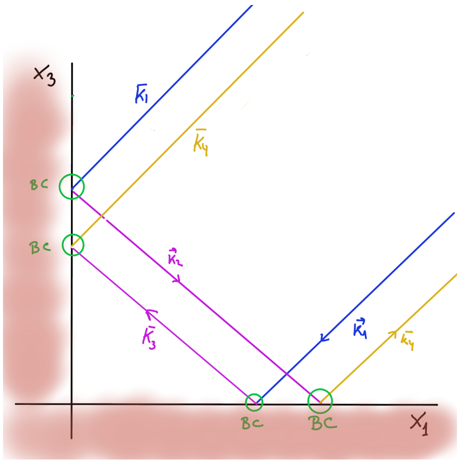


Figure 21: Our bouncy wave.

Let us assume that $\mathbf{E}_i$ to be polarised in $\hat{x}_2$ direction. Their wave vectors have the following form:

$$\mathbf{k}_i = \mp k_1 \hat{x}_1 \pm k_3 \hat{x}_3 + k_2 \hat{x}_2. \tag{3.4.58}$$

Now we have the crucial point; We will have to evaluate (**??**) for *each vertex*[17] in our

[17]Each vertex will be the LC of ingoing and outgoing waves respect to that plane.

system. This is:

$$
\begin{aligned}
&[(\mathbf{E}_1+\mathbf{E}_2)\times\hat{n}]_{x_1=0}=0 &&\rightarrow && E_1=-E_2,\\
&[(\mathbf{E}_3+\mathbf{E}_4)\times\hat{n}]_{x_1=0}=0 &&\rightarrow && E_3=-E_4,\\
&[(\mathbf{E}_1+\mathbf{E}_3)\times\hat{n}]_{x_3=0}=0 &&\rightarrow && E_1=-E_3,\\
&[(\mathbf{E}_3+\mathbf{E}_4)\times\hat{n}]_{x_3=0}=0 &&\rightarrow && E_2=-E_4.
\end{aligned}
\tag{3.4.59}
$$

It is important to notice that $\hat{n}$ will be different depending on which surface we evaluate in. Previous results yields how the amplitudes are related for each wave. Putting that information together in the linear combination for these fields, we get:

$$
\begin{aligned}
\mathbf{E}_T &= \sum_i \mathbf{E}_i,\\
&= E_0\hat{x}_2\left(e^{i(-k_1x_1-k_3x_3)}-e^{i(k_1x_1-k_3x_3)}-e^{i(-k_1x_1+k_3x_3)}-e^{i(k_1x_1+k_3x_3)}\right)e^{i(k_2x_2-\omega t)},\\
&= 4E_0\hat{x}_2 e^{i(k_2x_2-\omega t)}\sin(k_1x_1)\cos(k_3x_3)\,.
\end{aligned}
\tag{3.4.60}
$$

3.4.6 E: Waving at the Properties of a Wave

UNDER CONSTRUCTION

3.5 Waveguides and Cavities

3.5.1 Electromagnetic Crosswalk

We have the following distribution of beams in the sketched intersection:

$$\vec{E}_H = -E_0 e^{i(kx-\omega t)}\hat{z} \tag{3.5.1a}$$

$$c\vec{B}_H = E_0 e^{i(kx-\omega t)}\hat{y} \tag{3.5.1b}$$

$$\vec{E}_V = E_0 e^{i(ky-\omega t)}\hat{z} \tag{3.5.1c}$$

$$c\vec{B}_V = E_0 e^{i(ky-\omega t)}\hat{x} \tag{3.5.1d}$$

a)

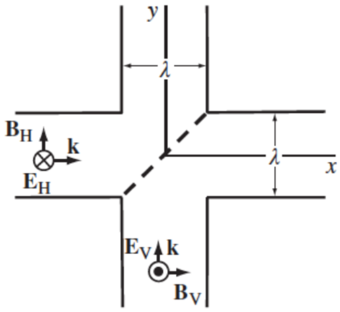


Figure 22: A sketch representation of the crossing beams and their components.

The H and V beams are both propagating, monochromatic plane waves with electric field amplitude E_0. We know that the time-average density energy is given by:

$$\langle u_{\text{EM}}\rangle_{H,V} = \text{Re}\left\{\frac{\epsilon_0}{4}\left(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*\right)\right\} = \frac{1}{2}\epsilon_0 E_0^2. \tag{3.5.2}$$

But at the very core of the intersection we are going to have a linear combination of $\mathbf{E}_H + \mathbf{E}_V$ and $\mathbf{B}_H + \mathbf{B}_V$, so this requires some computation to clean the following expression:

$$\begin{aligned}\langle u_{\text{EM}}\rangle_{\text{Total}} &= \text{Re}\left\{\frac{\epsilon_0}{4}\left[(\mathbf{E}_H + \mathbf{E}_V)\cdot\left(\mathbf{E}_H^* + \mathbf{E}_V^*\right) + c^2(\mathbf{B}_H + \mathbf{B}_H)\cdot\left(\mathbf{B}_H^* + \mathbf{B}_V^*\right)\right]\right\} = \\ &= \text{Re}\left\{\frac{1}{4}\left(\epsilon_0\left(\mathbf{E}_H\mathbf{E}_H^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_H\mathbf{B}_H^*\right)\right\} + \text{Re}\left\{\frac{\epsilon_0}{4}\left(\mathbf{E}_V\mathbf{E}_V^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_V\mathbf{B}_V^*\right)\right)\right\} + \\ &\quad + \text{Re}\left\{\frac{\epsilon_0}{4}\left(\mathbf{E}_H\mathbf{E}_V^* + \mathbf{E}_V\mathbf{E}_H^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_H\mathbf{B}_V^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_V\mathbf{B}_H^*\right)\right\} = \\ &= \langle u_{\text{EM}}\rangle_H + \langle u_{\text{EM}}\rangle_V + \\ &\quad + \text{Re}\left\{\frac{\epsilon_0}{4}\left(\mathbf{E}_H\mathbf{E}_V^* + \mathbf{E}_V\mathbf{E}_H^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_H\mathbf{B}_V^* + \tfrac{1}{\epsilon_0\mu_0}\mathbf{B}_V\mathbf{B}_H^*\right)\right\}.\end{aligned} \tag{3.5.3}$$

Let us study some of the pieces of the last term in the previous line, so we can simplify even further. Observe that $\mathbf{B}_H\mathbf{B}_V^* = \mathbf{B}_V\mathbf{B}_H^* = 0$ because $\hat{y}\cdot\hat{x} = 0$. So the only possible contribution comes from:

$$\mathbf{E}_H\mathbf{E}_V^* = -E_0^2\, e^{i(kx-ky)}\underbrace{\hat{z}\cdot\hat{z}}_{=1}. \tag{3.5.4}$$

And similarly for the other term with opposite sign in the exponent. So puting all together and recalling that $\langle u_{\rm EM}\rangle_{H,V} = \frac{1}{2}\epsilon_0 E_0^2$, we have:

$$\begin{aligned}\langle u_{\rm EM}\rangle_{Total} &= \tfrac{1}{2}\epsilon_0 E_0^2 + \tfrac{1}{2}\epsilon_0 E_0^2 - \tfrac{1}{2}\epsilon_0 E_0^2\,\mathrm{Re}\underbrace{\left\{e^{i(kx-ky)} + e^{-i(kx-ky)}\right\}}_{=2\cos(\mathrm{arg})} = \\ &= \tfrac{1}{2}\epsilon_0 E_0^2\left[2-\cos(k\left[x-y\right])\right]\end{aligned} \tag{3.5.5}$$

This quantity is minimum when $x = y$. On that plane (the diagonal of the crosswalk), the physical fields are:

$$\mathbf{E}(x=y) = 0\hat{\mathbf{z}} \quad \text{and} \quad c\mathbf{B}(x=y) = E_0\cos(kx-\omega t)(\hat{\mathbf{x}}+\hat{\mathbf{y}}) \tag{3.5.6}$$

b)

Now we are asked to do exactly the same for the time-averaged Poynting vector for a plane wave propagating in the $\hat{\mathbf{k}}$ direction. This expression reads:

$$\langle\mathbf{S}\rangle = \sqrt{\tfrac{\epsilon_0}{\mu_0}}|E_0|^2\hat{k} = c\,\langle u_{\rm EM}\rangle\,\hat{\mathbf{k}}. \tag{3.5.7}$$

Therefore, the horizontal and vertical beams poynting expression follow from the previous part of the exercise as:

$$\langle\mathbf{S}\rangle_V = c\epsilon_0 E_0^2\hat{\mathbf{y}} \quad \text{and} \quad \langle\mathbf{S}\rangle_H = c\epsilon_0 E_0^2\hat{\mathbf{x}} \tag{3.5.8}$$

So the superposition reads:

$$\begin{aligned}\langle\mathbf{S}\rangle &= \frac{1}{\mu_0}\mathrm{Re}\left\{(\mathbf{E}_H+\mathbf{E}_V)\times(\mathbf{B}_H+\mathbf{B}_V)^*\right\} = \\ &= \langle\mathbf{S}_H\rangle + \langle\mathbf{S}_V\rangle + \frac{1}{\mu_0}\mathrm{Re}\left\{\mathbf{E}_H\times\mathbf{B}_V^* + \mathbf{E}_V\times\mathbf{B}_H^*\right\} = \\ &= 2\,\epsilon_0\,c\,E_0^2(\hat{\mathbf{x}}+\hat{\mathbf{y}}) - \frac{E_0^2}{\mu_0}\mathrm{Re}\underbrace{\left\{e^{ik(x-y)}\hat{\mathbf{y}} + e^{ik(y-x)}\hat{\mathbf{x}}\right\}}_{2\cos(\mathrm{arg})} = \\ &= 2\,\epsilon_0\,cE_0^2[1-\cos k(x-y)](\hat{\mathbf{x}}+\hat{\mathbf{y}})\end{aligned} \tag{3.5.9}$$

From this previous expression we can see that, again, something interesting will take place when $x = y$. Observe the following sketch. The Poynting vector grows outwards from the diagonal $x = y$.

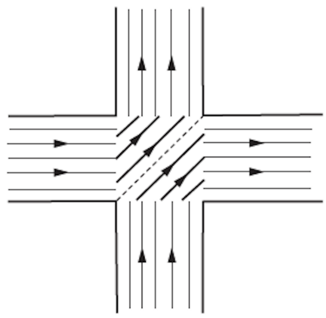


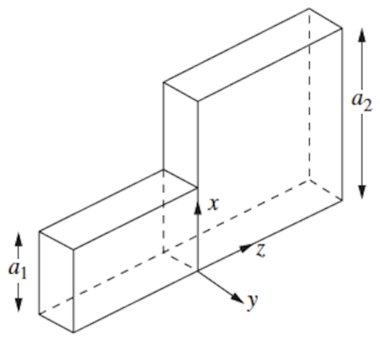
Figure 23: The Poynting vector is nule in the $x = y$ diagonal.

c)

At the surface of a conductor, we must have $\mathbf{E}_{\parallel} = 0$ and $\mathbf{B}_{\perp} = 0$. For this problem, part (a) shows that $\mathbf{E}$ points along $\hat{\mathbf{z}}$ and goes to zero at $x = y$. Part (a) showed also that B is parallel to the $x = y$ plane everywhere in the overlap region. Hence, the boundary conditions for a perfect conductor are met at $x = y$.

3.5.2 Waveguide Discontinuity

The first question one has to ask is how the set-up looks like. It results to be something like the following:



Observe that $a_2 > a_1$. The wave we want to study is moving from the region of a_1 to a_2. That difference of space will affect the modes of the wave, but how?

Recall that a TE-mode means that $E_{\parallel} = 0$ respect to the motion of the wave. This also implies that $\frac{\partial B_z}{\partial n}|_s = 0$. For the subscript 1,0 we refer to the specific mode of the solution we want to study. A general wave solution for a system like this goes as:

$$\psi_1(x, y) = H_{m,n} \cos\left(\frac{m\pi x}{a_1}\right) \cos\left(\frac{n\pi y}{b}\right). \tag{3.5.10}$$

For a $\text{TE}_{m,0}$ mode in the first region, we have $\psi_1 \propto \cos(m\pi x/a_1)$. As there is this change in height in the waveguide between the first and the second region, we do not know which new modes can or not appear. This can expressed as the following:

$$\psi_2(x, y) = \sum_m \psi_{m,0} = \sum_m H_{m,0} \cos\left(\frac{m\pi x}{a_2}\right). \tag{3.5.11}$$

The continuity of the tangential component of E shows that only TE_{m0} modes will propagate in waveguide 2 because the absence of y -dependence in guide 1 cannot generate y -dependence in guide 2 . Our task, then, is to find the expansion coefficients $H_{m,0}$ so

$$\psi_2 = \sum_{m=1}^{\infty} H_{m,0} \cos\left(\frac{m\pi x}{a_2}\right) = \begin{cases} H\cos(\pi x/a_1), & 0 \le x \le a_1 \\ 0, & a_1 < x \le a_2 \end{cases} \tag{3.5.12}$$

where we refer to $H_{1,0}$ as H. This is a (quite dirty) job for the orthogonality properties of the cosine functions. Integrating we arrive to:

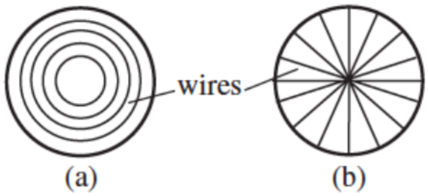
$$\begin{aligned} H_{m,0} &= \frac{2H}{a_1} \int_0^{a_1} dx \cos(\pi x/a_1) \cos(m\pi x/a_2) = \\ &= \frac{2Ha_2}{\pi(a_2 - ma_1)} \sin[\pi(1 - ma_1/a_2)] - \frac{2Ha_2}{\pi(a_2 + ma_1)} \sin[\pi(1 + ma_1/a_2)] = \\ &= \frac{2Hma_1a_2}{\pi\left(a_2^2 - m^2a_1^2\right)} \sin\left(m\pi\frac{a_1}{a_2}\right). \end{aligned} \tag{3.5.13}$$

When $a_1 = a_2, H_{m,0} = 0$ for $m \neq 1$. When $m = 1$, l'Hospital's rule gives the expected answer,

$$H_{1,0} = \lim_{m\to 1} \frac{d}{dm} \frac{2H}{\pi} \frac{\sin(m\pi)}{\left(1 - m^2\right)} = \lim_{m\to 1} \frac{2\pi H \cos(m\pi)}{-2m\pi} = H. \tag{3.5.14}$$

3.5.3 Guess Who? (Wavefilter Edition)

Let first display how this two filters look like.



So we know that one has a TM- mode and the other one has a TE. Which is which?

Let's start from the basics. A general wave-guide will relate the **E** and **B** fields as:

$$\vec{\nabla} \times \mathbf{E}_{\perp} = i\omega B_z \hat{z}. \tag{3.5.15}$$

And the modes requirements imply:

$$\begin{aligned} \text{TM} \rightarrow B_z = 0 \quad \text{and} \quad & E_z|_{\text{surface}} = 0, \\ \text{TE} \rightarrow E_z = 0 \quad \text{and} \quad & \tfrac{\partial B_z}{\partial n}|_{\text{surface}} = 0. \end{aligned} \tag{3.5.16}$$

In order to determine which mode correspond to which tube, let us choose the behavior of a TM mode, which implies then:

$$\vec{\nabla} \times \mathbf{E}_{\perp} = 0 \quad \rightarrow \quad \vec{\nabla} \times \vec{\nabla} \cdot \phi = \vec{\nabla} \times (\partial_x \phi, \partial_y \phi, 0). \tag{3.5.17}$$

This is giving us a hint. If we think in terms of small differences of the field ϕ in the x, y directions, we see that there should not be any change. Hence, we can discard the possibility of this mode to be contained along those radial wires, which break the symmetry. This is not the case of the tube A, where $\Delta E|_{r=\text{cnt}} = 0$. Then, TM corresponds to the tube A while TE mode corresponds to the tube B.

3.5.4 An Electromagnetic Bat in a Resonant Cavity

In order to succesfully solve this problem we have to realise several things. The first one is that the combination of waves can be expressed as a linear combination of them as:

$$\psi(x,y,t) = \sum_{m=0}^{n} (-1)^m \sin(\mathbf{k}_m \cdot \mathbf{r} - ckt), \tag{3.5.18}$$

Where n is the total number of waves. The second thing to realise is that TM modes in a cavity have the property that $\psi = 0$ on the walls of the cavity. Instead of getting a reflection of the wave we emit[18], we can sketch the inner structure of the cavity by using this anhilition property of the waves. We have to find the explicit expression of the L.C. of waves we have inside de cavity and solve a system of equations to determine the boundaries of this space. For that we need to compute $\mathbf{k}_i \cdot \mathbf{r}$.

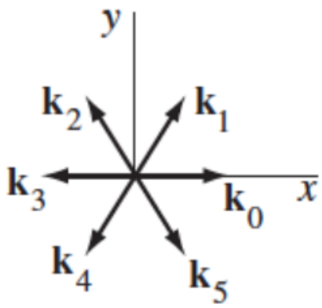


Figure 24: The vectorial distribution of the six waves.

$$\begin{aligned}
\mathbf{k}_0 \cdot \mathbf{r} &= kx, \\
\mathbf{k}_1 \cdot \mathbf{r} &= \cos\left[\frac{\pi}{3}\right] kx + \sin\left[\frac{\pi}{3}\right] ky = \frac{1}{2}kx + \frac{\sqrt{3}}{2}ky, \\
\mathbf{k}_2 \cdot \mathbf{r} &= \cos\left[\frac{2\pi}{3}\right] kx + \sin\left[\frac{2\pi}{3}\right] ky = -\frac{1}{2}kx + \frac{\sqrt{3}}{2}ky, \\
\mathbf{k}_3 \cdot \mathbf{r} &= -kx, \\
\mathbf{k}_4 \cdot \mathbf{r} &= \cos\left[\frac{4\pi}{3}\right] kx + \sin\left[\frac{4\pi}{3}\right] ky = -\frac{1}{2}kx - \frac{\sqrt{3}}{2}ky, \\
\mathbf{k}_5 \cdot \mathbf{r} &= \cos\left[\frac{5\pi}{3}\right] kx + \sin\left[\frac{5\pi}{3}\right] ky = \frac{1}{2}kx - \frac{\sqrt{3}}{2}ky.
\end{aligned} \tag{3.5.19}$$

So basically, we have a reflection of $\{0,1,2\}$ for $\{3,4,5\}$ waves. Expanding ψ we see:

[18] radar technology.

$$
\begin{aligned}
\psi(x,y,t) =& \sin(\mathbf{k}_0\cdot\mathbf{r}-ckt)-\sin(\mathbf{k}_1\cdot\mathbf{r}-ckt)+\sin(\mathbf{k}_2\cdot\mathbf{r}-ckt)-\\
&-\sin(\mathbf{k}_3\cdot\mathbf{r}-ckt)+\sin(\mathbf{k}_4\cdot\mathbf{r}-ckt)-\sin(\mathbf{k}_5\cdot\mathbf{r}-ckt)=\\
=&\mathbf{Im}\left\{e^{-i\omega t}\left(e^{i(\mathbf{k}_0-\mathbf{k}_3)\mathbf{r}}-e^{i(\mathbf{k}_1-\mathbf{k}_4)\mathbf{r}}+e^{i(\mathbf{k}_2-\mathbf{k}_5)\mathbf{r}}\right)\right\}=\\
=&2\cos(\omega t)\left\{\sin kx-\sin\left[\frac{kx}{2}+\frac{\sqrt{3}ky}{2}\right]-\sin\left[\frac{kx}{2}-\frac{\sqrt{3}ky}{2}\right]\right\}.
\end{aligned}
\tag{3.5.20}
$$

Now we have an explicit expression for ψ. It is time to look for its 0's. To simplify our life, we can use the following for the last to sin.

$$
\begin{aligned}
\sin(a+a)-\sin(a+b)-\sin(a-b) =&[\sin a\cos a+\cos a\sin a]-\\
&-[\sin a\cos b+\cos a\sin b]-\\
&-[\sin a\cos b-\cos a\sin b]=\\
=&2\sin a[\cos a-\cos b].
\end{aligned}
\tag{3.5.21}
$$

So the zeroes will be sitting at:

$$
\psi(x,y,t)=2\cos(wt)\left\{2\left(\sin(\tfrac{kx}{2})\left(\cos(\tfrac{kx}{2})-\cos(\tfrac{\sqrt{3}ky}{2})\right)\right)\right\}
\tag{3.5.22}
$$

Where we have used the trick $\sin(kx)=2\sin(kx/2)\cos(kx/2)$. This implies that we have two possibilities, such that:

$$
\begin{aligned}
\sin(\tfrac{kx}{2})=0 \quad &\rightarrow \quad x=\tfrac{2n\pi}{k},\\
\cos(\tfrac{kx}{2})-\cos(\tfrac{\sqrt{3}ky}{2})=0 \quad &\rightarrow \quad x=\pm\sqrt{3}y.
\end{aligned}
\tag{3.5.23}
$$

Therefore, if $\lambda=2\pi/k$, the heavy solid lines in the figure below outline a 2D conducting cavity which will support a TM resonant mode built from $\psi(x,y,t)$.

3.5.5 E: Rectangular Waveguide and its Modes

UNDER CONSTRUCTION

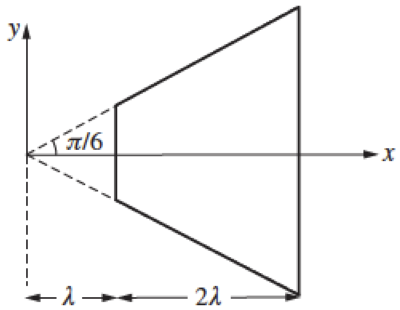


Figure 25: Our electromagnetic bat is safe between these four walls.

3.5.6 E: Mirror mirror on the wall...

1):

The boundary conditions that both fields have to satisfy on the surface of a perfect conductor are just:

$$\mathbf{E}_{\parallel} = 0, \qquad \mathbf{B}_{\perp} = 0. \tag{3.5.24}$$

2):

We know that we have to find a solution to:

$$\left(\nabla_{\perp}^2 + \gamma^2\right)\Psi = 0, \tag{3.5.25}$$

that we have to solve for the each of the modes (a.k.a different boundary solutions). We already know that sin and cos are eigenfunctions for ∂_i^2 operator. This will help us to construct the form of the solution, as we will impose the field boundary conditions given by the conductor wall properties to determine if we need one trigonometric function or the other.

TM modes

We have $E_z(x, y) = 0$ at the boundaries. Also, we want the parallel component of $\mathbf{E} = 0$ when $x = y = 0$, so when we are in one of the corners. cos is not a 0 at this corner, so our solution to (**??**) looks like:

$$E_z(x, y) = E_0 \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{a}\right). \tag{3.5.26}$$

This will give a cut-off of the form:

$$\gamma^2 = \frac{\pi^2}{a^2}\left(n^2 + m^2\right). \tag{3.5.27}$$

TE modes

Same story here. Now, the boundary condition is that $\hat{n}\cdot\vec{\nabla}_T B^z(x,y) = 0$ at the walls of the conductor. So we require the cos function as:

$$B_z(x,y) = B_0 \cos\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi m y}{a}\right). \tag{3.5.28}$$

With exactly the same cut-off frecuency γ^2 as before.

3):

Why do different modes have the same cut-off frequency? Just take a look at expression (**??**). If one interchanges n, m values, the result is the same, but not the modes of $\mathbf{B}_{n,m}$ and $\mathbf{B}_{m,n}$. So basically, there is a symmetry in expression (**??**).

Actually, we can exploit this symmetry to craft modes that can be allowed in the second scenario, when one cuts the waveguide through the diagonal. In this case we want to impose that the fields, depending on the modes we study, are 0 in the line $x + y = a$.

TM modes

Here we need to impose that $E_z = 0$ on that line. Let's blindly follow the statement and generate a general linear combination with both possibilities.

$$E_z = E_0^{(1)} \sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{a}\right) + E_0^2 \sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{a}\right). \tag{3.5.29}$$

We know that this expression should be equal to 0 when $y = a - x$. If we evaluate there and use double angle trigonometric identities in wise way, we arrive to the following expression:

$$\begin{aligned} 0 = E_z|_{y=a-x} = E_0^{(1)} \sin\left(\frac{\pi n x}{a}\right)(-1)^{m+1}\sin\left(\frac{\pi m x}{a}\right) + E_0^2 \sin\left(\frac{\pi m x}{a}\right)(-1)^{n+1}\sin\left(\frac{\pi n x}{a}\right) \rightarrow \\ E^{(2)} = (-1)^{m+n+1} E_0^{(1)}. \end{aligned} \tag{3.5.30}$$

So the expression in this case for a triangular waveguide is:

$$E_z = E_0\left(\sin\left(\frac{\pi n x}{a}\right)\sin\left(\frac{\pi m y}{a}\right) + (-1)^{n+m+1}\sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n y}{a}\right)\right). \tag{3.5.31}$$

TE modes

Same story here. In this case the conditions will come from $(\partial_x + \partial_y)B_z = 0$. If we craft a general linear combination for the expression of the magnetic field as we did in (**??**), compute its derivates and plug those requirements into this linear combination expression, we will get:

$$B_z = B_0\left(\cos\left(\frac{\pi n x}{a}\right)\cos\left(\frac{\pi m y}{a}\right) + (-1)^{n+m}\cos\left(\frac{\pi m x}{a}\right)\cos\left(\frac{\pi n y}{a}\right)\right). \tag{3.5.32}$$

Powerful symmetry to exploit, is it not?

3.6 Radiation and Scattering

3.6.1 Electric Dipole Radiation

As always, the first we should do is to sketch the system we want to study;

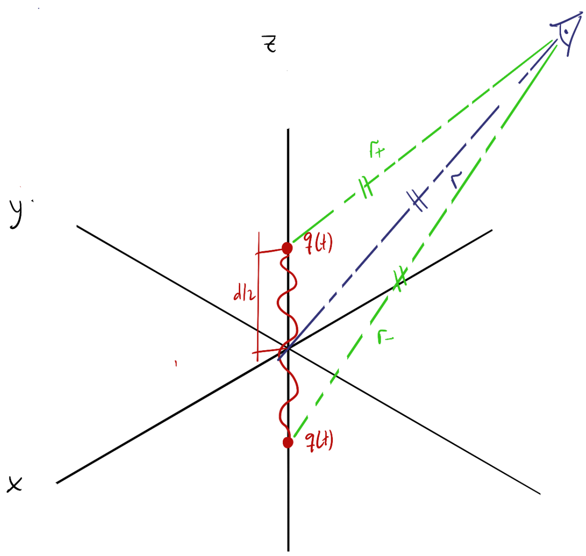


Figure 26: Our time dependant dipole.

Where $q(t) = q_0 \cos\omega t$. We can compute the potential as a superposition of both charges as:

$$V(r) = \sum_{i=0}^{n} \frac{q_i}{4\pi\epsilon_0 r_i} = \frac{q(t)}{4\pi\epsilon_0 r_1} - \frac{q(t)}{4\pi\epsilon_0 r_2}. \tag{3.6.1}$$

As we can see in the sketch, there is a difference in the paths r_i. If we are *Far Far away*[19], we can always approximate $r \gg \frac{d}{2}$, which will allow us to expand the norm of $\vec{r}_i$ as:

$$|\vec{r}_i| = \sqrt{(\vec{r} \pm (0,0,\frac{d}{2}))^2} \approx \sqrt{r^2 + \frac{d^2}{4} \pm dr\cos\theta}. \tag{3.6.2}$$

But one has also to consider that the change of charge in r position will have a different delay depending on the path they have to travel. In this case we have to take into account the retarded time as:

$$t_\pm = t - \frac{r_\pm}{c}. \tag{3.6.3}$$

So the potential gets more involved with the following appearance:

$$V(r) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos(\omega(t-\frac{r_+}{c}))}{\sqrt{r^2+\frac{d^2}{4}+dr\cos\theta}} - \frac{\cos(\omega(t-\frac{r_-}{c}))}{\sqrt{r^2+\frac{d^2}{4}-dr\cos\theta}}\right]. \tag{3.6.4}$$

And it is now when we can start taking radiation approximations.

1):

If we want to compute the potential far away, it basically means that $\frac{d}{r} \to 0$. Extract r in the denominator of expression (**??**) to see:

$$V(r) = \frac{q_0}{4\pi\epsilon_0}\left[\frac{\cos(\omega(t-\frac{r_+}{c}))}{r\sqrt{1+\cancel{\frac{d^2}{4r^2}}+\frac{d}{r}\cos\theta}} - \frac{\cos(\omega(t-\frac{r_-}{c}))}{r\sqrt{1+\cancel{\frac{d^2}{4r^2}}-\frac{d}{r}\cos\theta}}\right]. \tag{3.6.5}$$

This will also affect $r_\pm$ inside cos, where we have to Taylor expand the square root piece, yielding:

[19]With Fiona and her parents.

$$\cos\left(\omega\left(t-\frac{r_\pm}{c}\right)\right) \simeq \cos\left(\underbrace{\omega\left(t-\frac{r}{c}\right)}_{A} \pm \underbrace{\frac{\omega r d}{2rc}\cos\theta}_{B}\right) =$$

$$\to \frac{c}{w} \gg d \to \text{sin and cos with argument } \frac{d}{c/w} \text{ can be Taylor expanded} \to \tag{3.6.6}$$

$$= \cos\left(\omega\left(t-\frac{r}{c}\right)\right) \times 1 \mp \sin\left(\omega\left(t-\frac{r}{c}\right)\right) \times \frac{\omega d}{2c}\cos\theta.$$

It is important to notice that double angle $\cos(A+B)$ has been expanded and then, leading contributions of the Taylor expansion have been taken (Hence $\cos(x) \sim 1$ and $\sin(x) \sim x$). Realise also that the term $\frac{d}{r}$ inside the square roots of denominators in (**??**) is small compared to 1. This has to be approximated by a Taylor series. Computing this, the potential V looks like (to 1^{st} order):

$$V(r) = \frac{q_0 d\cos\theta}{4\pi\epsilon_0 r}\left[\frac{-\sin(\omega(t-\frac{r}{c}))}{c}\omega + \frac{\cos(\omega(t-\frac{r}{c}))}{r}\right]. \tag{3.6.7}$$

2):

In the case we want to discuss what happens if $\omega \to 0$ we just have to realise the behaviour of the trigonometric functions when its argument is 0. In this case:

$$V(r) = \frac{d q_0 \cos\theta}{4\pi\epsilon_0 r^2}. \tag{3.6.8}$$

Which is basically a non-vibrating potential, as expected.

3):

If we want to simplify more the result obtained in section 1), demanding that the observation distances is way bigger than the emitted wavelength, we just have to study the schematic form the potential as:

$$V(r) \propto \frac{A}{r} \times \frac{c}{\omega} + \cancel{\frac{B}{r^2}}. \tag{3.6.9}$$

We can drop the second term as it goes with the inverse of the square of r, which will yield no contribution the further we observe. This can be done as we know that $\cos \in [-1,1]$.

4):

To compute the vector potential $\vec{A}(t,\vec{x})$ given the assumptions we have been working with during the previous sections, we just have to integrate the following under some considerations:

$$\vec{A}(t,\vec{x}) = \tfrac{\mu_0}{4\pi}\int d^3x' \tfrac{1}{R}\vec{J}(t',\vec{x}')_{ret}. \tag{3.6.10}$$

Where $\vec{J}$ is given by the charge flow.[20] As we have the dipole oriented in the z-direction, this is where the current will be located, as the variation in time of the charge, so:

$$\vec{J} = \tfrac{dq}{dt}\hat{v}_{\text{flow}} = -q_0\omega\sin\left(\omega(t-\tfrac{r_\pm}{c})\right). \tag{3.6.11}$$

And the integration limits are given by the position of each of the charges (From now on, we can imagine this set-up as a small antenna located along the z-axis), so $z \in [-\frac{d}{2},\frac{d}{2}]$.In this specific case we do not have to care too much about r_{pm}, as its change in value [21] will not change. So integral results, after all in:

$$\vec{A}(t,\vec{x}) = \frac{\mu_o}{4\pi}\frac{q_0\omega d\sin\left(\omega(t-\frac{r}{c})\right)}{r}\hat{z}. \tag{3.6.12}$$

5):

To finish this exercise, we are going to compute **E** and **B** in the mentioned limits. To simplify our calculations, lets move our coordinate system from Cartesian to spherical coordinates, just by:

$$\hat{z} \to (\cos\theta, -\sin\theta, 0). \tag{3.6.13}$$

Then, we just have to finally compute **E** and **B** using Maxwell's equations. Extracting from our memory[22] the expression of the rotational and gradient in spherical coor-

[20] Other option is to use the continuity equation to obtain $\vec{J}$. The charge are discrete ones located at $[-\frac{d}{2},\frac{d}{2}]$, so the integration should be easy, just in the z direction.
[21] if we moved dz a little bit through the axis, as we are far away from the source, the value of r_{pm} would not change too much, allowing us to identifying it as r.
[22] or Wikipedia.

dinates[23] we can start computing them. For **B** we find:

$$\mathbf{B} = \frac{\mu_0 q_0 \omega d \sin\theta}{4\pi r}\left(\tfrac{-\omega}{c}\cos\left(\omega(t-r/c)\right) + \tfrac{1}{r}\sin\left(\omega(t-r/c)\right)\right)\hat{\phi}. \tag{3.6.14}$$

In the case we want to compute the electric field, we have to do two steps (as it has contributions from the electric potential V and the vector potential $\vec{A}$). Again, observe there is no dependence on ϕ, which simplifies our calculations. Also, to simplify the calculation of the gradient of V, we just have to know that Taylor expansions and derivates commute. This means one can take expression (**??**) and gradient it. At the end of the day, summing carefully, this yields:

$$\begin{aligned}\mathbf{E} = &-\left(\frac{d q_0}{4\pi\omega\epsilon_0 r^2}\cos\theta\sin\left(\omega(t-r/c)\right)\right)\hat{r}- \\ &-\left(\frac{d q_0}{4\pi\omega\epsilon_0 r^2}\sin\theta\sin\left(\omega(t-r/c)\right) + \frac{d q_0\mu_0\omega^2}{4\pi r}\sin\theta\cos\left(\omega(t-r/c)\right)\right)\hat{\theta}.\end{aligned} \tag{3.6.15}$$

We may think we are done, but we are mistaken. Recall from section 3, that if we want to simplify our calculations by observing from far away, this implies that terms $\propto \frac{1}{r^2}$ will barely have contribution. This means, that after travelling to far, far away through this problem, we arrive to the final result:

$$\begin{aligned}\mathbf{E} &= \left(0, -\frac{d q_0\mu_0\omega^2}{4\pi r}\sin\theta\cos\left(\omega(t-r/c)\right), 0\right), \\ \mathbf{B} &= \left(0, 0, \frac{-\mu_0 q_0\omega^2 d\sin\theta}{4\pi c r}\cos\left(\omega(t-r/c)\right)\right).\end{aligned} \tag{3.6.16}$$

3.6.2 Metallic Shells

As always, the first step to find a solution to this problem is to sketch the current set-up, which looks:

One can derive the potential for this system using Green functions and proper boundary conditions. Also, one can find an explicit expression for this potential in Jackson (section 2.7, pg. 64), which is:

[23] Observe that there is no dependence on ϕ.

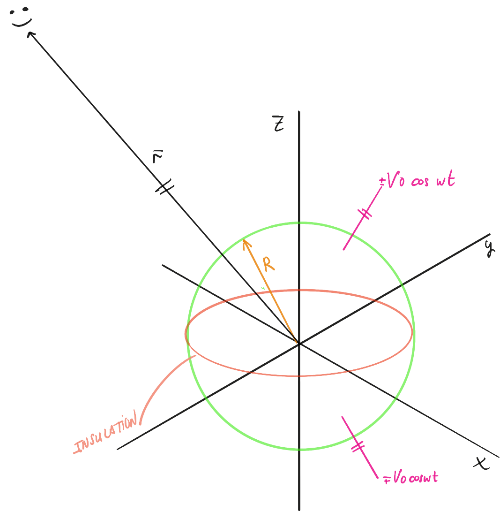


Figure 27: The metallic shells with oscillating potentials.

$$\Phi(r,\theta,\phi) = \frac{3V_{\pm}R^2}{2r^2}\left(\frac{r^3(r^2-R^2)}{(r^2+R^2)^{\frac{5}{2}}}\right)\cos\theta\cdots \tag{3.6.17}$$

Where $\cdots$ mean that there are higher order corrections. Recall that r is the observation position, R stands for the radius of the shells and $V_{\pm}$ is a short-hand notation for $V_0\cos\omega t$.

If we are in the long-wavelength limit ($r \gg \omega \gg R$), it also means that we have $r \gg R$, so we can simplify previous expression by:

$$\begin{aligned}\Phi(r,\theta,\phi)|_{r\gg R} &= \frac{3V_{\pm}R^2}{2r^2}\left(\frac{r^3(r^2-\cancel{R^2})}{(r^2+\cancel{R^2})^{\frac{5}{2}}}\right)\cos\theta\cdots = \\ &= \frac{\pm 3V_0\cos\omega t\, R^2}{2r^2}\cos\theta.\end{aligned} \tag{3.6.18}$$

So we now have an appropriated approximation of the potential for this system. Then we can start computing the electric dipole. At this stage, we can exploit the fact that the potential generated by a regular dipole is:

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left(\sum_i \frac{q_i}{r} + \sum_i \frac{\vec{p}_i \cdot \vec{x}_i}{r^3} + \cdots\right). \tag{3.6.19}$$

So, if we think in terms of being far, far away, we can think of the metallic shell system with oscillating potentials as dipole generated by two "charges" at the origin. But here we have no charges! (So the first term in (**??**) is 0). If we compare (**??**) with the simplified potential in the long wavelength limit, we find:

$$\begin{aligned}\Phi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\left(\frac{p\, r\, \cos\theta}{r^3}\right) = \frac{3V_{\pm}R^2}{2r^2}\cos\theta \rightarrow \\ \vec{p} = 6\pi V_{\pm}R^2\hat{z}.\end{aligned} \tag{3.6.20}$$

Where, keeping the analogy with the dipole generated by two charges at the origin, we can ensure a $\hat{z}$ orientation of this one. This will be the first step to obtain the radiation field and the radiated power.

$\mathbf{E},\mathbf{H}$ fields of a radiating system are given by:

$$\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r}\left(1+\frac{1}{ikr}\right),\\ \mathbf{E} &= \frac{1}{4\pi\epsilon_0}\left(k^2(\vec{n}\times\vec{p})\times\vec{n}\,\frac{e^{ikr}}{r} + \left[3\vec{n}(\vec{n}\cdot\vec{p})-\vec{p}\right]\left(\tfrac{1}{r^3}-\tfrac{ik}{r^2}\right)e^{ikr}\right).\end{aligned} \tag{3.6.21}$$

But in the long wavelength limit, and far far away, these two expressions get simplified to ($\frac{1}{r^n}$ will barely contribute):

$$\begin{aligned}\mathbf{H} &= \frac{ck^2}{4\pi}(\vec{n}\times\vec{p})\frac{e^{ikr}}{r} = \cdots = \tfrac{3ck^2}{2r}\epsilon_0 VR^2 e^{ikr}\sin\theta\hat{\phi}\\ \mathbf{E} &= \frac{1}{c\epsilon_0}\mathbf{H}\times\vec{n} = \cdots = -\tfrac{3k^2}{2r}VR^2e^{ikr}\sin\theta\hat{\theta}.\end{aligned} \tag{3.6.22}$$

So with these two fields we just have to plug them inside the radiated power per solid angle to get:

$$\frac{dP}{d\Omega} = \tfrac{1}{2}\Re\left[r^2\vec{n}\cdot\mathbf{E}\times\mathbf{H}^*\right] = \cdots = \tfrac{9}{8}c\epsilon_0 k^4V^2R^4\sin^2\theta. \tag{3.6.23}$$

So we just have to integrate over the solid angle to get the total radiated power as:

$$\begin{aligned} P = \int d\Omega\, P = \underbrace{\tfrac{9}{8} c\epsilon_0 k^4 V^2 R^4}_{A} \int \sin^2\theta \sin\theta\, d\theta\, d\phi, \\ = \frac{3\pi}{2} c\epsilon_0 k^4 V^2 R^4. \end{aligned} \tag{3.6.24}$$

3.6.3 Electrostatic Potential from a Dipole

UNDER CONSTRUCTION

3.6.4 Radiation Interference

1):

We can infer, without knowing too much about radiation, that if we have two types of dipoles radiating simultaneously, some sort of interference will appear (may be constructive, destructive...). Since the two sources emit in phase,

$$\begin{aligned} \frac{dP}{d\Omega} &\propto |\hat{\mathbf{r}} \times (\alpha_{elec} + \alpha_{mag})|^2 = \\ &= (\hat{\mathbf{r}} \times \alpha_{elec})^2 + (\hat{\mathbf{r}} \times \alpha_{mag})^2 + 2\underbrace{(\hat{\mathbf{r}} \times \alpha_{elec})(\hat{\mathbf{r}} \times \alpha_{mag})}_{\text{interference term}} \end{aligned} \tag{3.6.25}$$

The interference term can be further simplified using basic vector identities and it results to be:

$$(\hat{\mathbf{r}} \times \mathbf{p}) \cdot [\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})] = (\hat{\mathbf{r}} \times \mathbf{p}) \cdot [\mathbf{m} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m})] = (\hat{\mathbf{r}} \times \mathbf{p}) \cdot \mathbf{m} = \hat{\mathbf{r}} \cdot (\mathbf{p} \times \mathbf{m}). \tag{3.6.26}$$

Which is 0 $\iff$ $\mathbf{p} \parallel \mathbf{m}$.

2):

If we want to check that the time-averaged total power emitted has no interference contribution, we have to integrate over the solid angle Ω.

$$
\begin{aligned}
P = \int d\Omega \frac{dP}{d\Omega} &\propto \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \hat{\mathbf{r}} \cdot (\mathbf{p} \times \mathbf{m}) \, \sin\theta \\
&\propto \hat{\mathbf{r}} \cdot (\mathbf{p} \times \mathbf{m}) \int_{-\pi/2}^{\pi/2} d\theta \sin\theta = 0.
\end{aligned} \tag{3.6.27}
$$

A zero as big as a cathedral.[24] No interference contribution over the whole spaubsce.

3.6.5 Sinusoidal thin Antenna

As always, let's start with a sketch of the system:

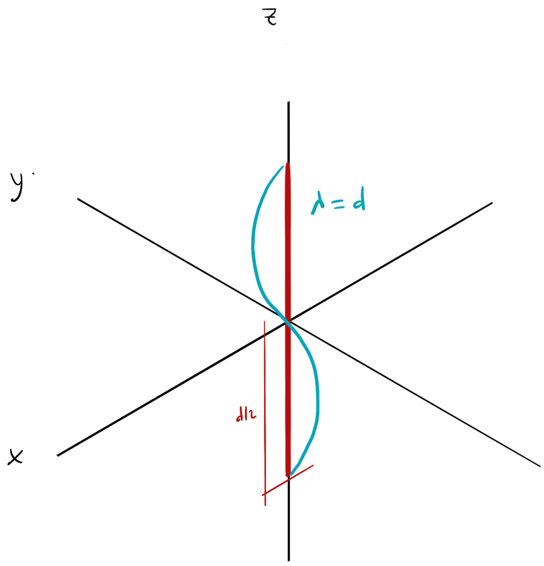


Figure 28: Our simple antenna sample.

a):

Note that the current flows in opposite directions in the top and bottom half of this antenna (It is born at the centre and goes up and down). As a result, we may write the source current density using the Heaviside distribution as:

$$
\vec{J}(z) = \hat{z} I \sin(kz)\delta(x)\delta(y)\Theta(d/2 - |z|), \qquad \text{with} \quad k = \frac{2\pi}{d}. \tag{3.6.28}
$$

In the radiation zone, the vector potential is given by the well know expression:

[24] Basic Spanish say when you want to specify that something is quite big/notorious.

$$
\begin{aligned}
\vec{A}(\vec{x}) &= \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int \vec{J}\left(\vec{x}'\right)e^{-ik\hat{n}\cdot\vec{x}'}d^3x', \\
&= \hat{z}\frac{\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_{-d/2}^{d/2}\sin(kz)e^{-ikz\cos\theta}dz.
\end{aligned}
\tag{3.6.29}
$$

At this point, when integrating, we have to possible options:

- To express the sin in terms of exponentials, rephrase, integrate and compose trigonometric functions again or...
- To perform a change of variables and use $\int_{-\pi}^{\pi}\sin(z)e^{-iaz}dz = \frac{2i\sin(\pi a)}{a^2-1}$.

All in all the result should be the same. Let's perform the first choice. Since the source current is odd under $z \to -z$, this integral may be written as:

$$
\begin{aligned}
\vec{A} &= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}2\sin(kz)\sin(kz\cos\theta)dz = \\
&= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{r}\int_0^{d/2}\left[\cos((1-\cos\theta)kz)-\cos((1+\cos\theta)kz)\right]dz = \\
&= -\hat{z}\frac{i\mu_0 I}{4\pi}\frac{e^{ikr}}{kr}\left[\frac{1}{1-\cos\theta}\sin((1-\cos\theta)kz)-\frac{1}{1+\cos\theta}\sin((1+\cos\theta)kz)\right]_0^{d/2} = \\
&= -\hat{z}\frac{i\mu_0 I}{2\pi}\frac{e^{ikr}}{kr}\frac{\sin(\pi\cos\theta)}{\sin^2\theta}.
\end{aligned}
\tag{3.6.30}
$$

To compute H in the radiation zone, recall that $\vec{\nabla} \sim ik\hat{n}$, so:

$$
\vec{H} = \frac{ik}{\mu_0}\hat{n}\times\vec{A} = -\hat{\phi}\frac{I}{2\pi}\frac{e^{ikr}}{r}\frac{\sin(\pi\cos\theta)}{\sin\theta}.
\tag{3.6.31}
$$

where we used $\hat{n}\times\hat{z} \equiv \hat{r}\times\hat{z} = -\hat{\phi}\sin\theta$. Then, the radiated power:

$$
\frac{dP}{d\Omega} = \frac{r^2}{2}\hat{n}\left(\vec{E}\times\vec{H}\right) = \underbrace{\frac{r^2}{2}\hat{n}\left(Z_0(\vec{H}\times\hat{n})\times\vec{H}\right)}_{\text{in the radiation zone.}}.
\tag{3.6.32}
$$

Using some identities, this results into a radiated power of the form:

$$\frac{dP}{d\Omega} = \frac{Z_0 r^2}{2}|\vec{H}|^2 = \frac{Z_0|I|^2}{8\pi^2}\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}. \tag{3.6.33}$$

b):

To compute the total radiated power we just have to integrate the previous result by the solid angle so:

$$P = \int_\Omega \frac{dP}{d\Omega}d\Omega = \frac{Z_0 I^2}{8}. \tag{3.6.34}$$

c):

This part of the problem is more pure calculations. We need the definition of the dipoles and quadropoles, given by:

$$\vec{p} = \int \vec{x}\,\rho\,d^3x, \qquad \vec{m} = \frac{1}{2}\int \vec{x}\times\vec{J}d^3x, \qquad Q_{ij} = \int\left(3x_i x_j - r^2\delta_{ij}\right)\rho d^3x. \tag{3.6.35}$$

To extract the charge density, we can use the equation of state $\vec{\nabla}\cdot J + \partial_t\rho = 0$. Also, recall that $\vec{J}(t,\vec{x}) = \vec{J}(\vec{x})e^{-i\omega t}$, so:

$$\rho = \frac{1}{i\omega}\vec{\nabla}\cdot\vec{J} = \frac{1}{i\omega}\frac{d\vec{J}}{dz} = -\frac{iI}{c}\cos(kz)\delta(x)\delta(y)\Theta(d/2-|z|) \tag{3.6.36}$$

where we used $\omega = ck$. The electric dipole moment is then:

$$\begin{aligned}\vec{p} = \int \vec{x}\rho d^3x = \int dx\,dy\,dz\,(x\hat{x}+y\hat{y}+z\hat{z})\,\frac{iI}{c}\cos(kz)\delta(x-0)\delta(y-0) = \\ = -\hat{z}\frac{iI}{c}\int_{-d/2}^{d/2} z\cos(kz)dz = 0\end{aligned} \tag{3.6.37}$$

Of course, a simple symmetry argument under $z\to -z$ demonstrates that this electric dipole must vanish. The magnetic dipole moment also vanishes since:

$$\vec{m} = \frac{1}{2}\int \vec{x}\times\vec{J}d^3x = \text{same story as before...} = \frac{I}{2}\int_{-d/2}^{d/2}\vec{z}\times[\hat{z}\sin(kz)]dz = 0 \quad (3.6.38)$$

We are left with an electric quadrupole moment as:

$$Q_{ij} = \int \left(3x_i x_j - r^2\delta_{ij}\right)\rho d^3x. \quad (3.6.39)$$

That gives us the expected $Q_{ij} = 0$ and:

$$\begin{aligned} Q_{xx} = Q_{yy} = -\tfrac{1}{2}Q_{zz} &= -\frac{I}{ic}\int_{-d/2}^{d/2}(3x_i^2 - \sum x_i^2)\delta(x)\delta(y)\cos(kz)dz = \\ &= \frac{-Id^3}{ic\pi^2}. \end{aligned} \quad (3.6.40)$$

3.6.6 Scattering in Solid Sphere

We first have to understand what the set-up. Let's observe the following sketch.

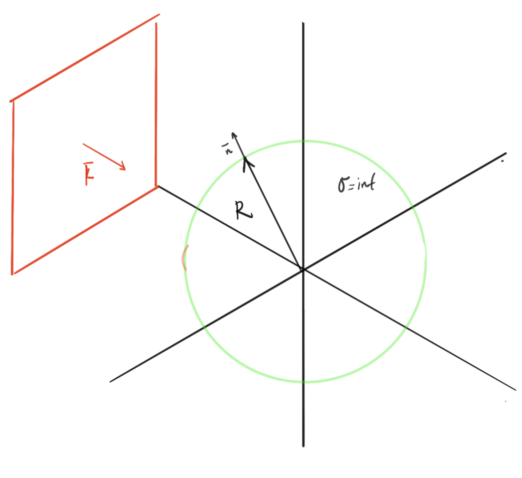


Figure 29: A sphere being hit by a plane wave.

Then, let's proceed with the different parts of this problem.

1):

Assuming that conductivity σ is infinite, we then just realise that there will be no fields inside the sphere, hence no current J inside this one. In the same spirit as in the electric case, one can think in terms of a "magnetic" potential ϕ_M, whose divergence will generate the magnetic field $\mathbf{B}$ around the sphere surface. In order to find this potential, let's use the multipole expansion of it and some boundary conditions. We know that the most general potential expansion looks like:

$$\Phi(\mathbf{r},\theta,\phi) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{3.6.41}$$

And what are the boundary conditions we know? That the potential on the surface should be 0 and $-B_0\, z$[25] when $r \gg R$. Then, the second boundary condition help us fix the first linear coefficient as:

$$\begin{aligned}
\phi_M|_{r\gg R} = -B_0 r\,\cos\theta &= \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \underbrace{\frac{B_\ell}{r^{\ell+1}}}_{r\gg R}\right)P_\ell(\cos\theta) =, \\
-B_0 r\,\cos\theta &= \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell\right)P_\ell(\cos\theta) = A_1\, r^1 \cos\theta =, \\
B_0 &= A_1.
\end{aligned} \tag{3.6.42}$$

There is a missing boundary condition that used now can shed some light on the sub-leading coefficient of this expansion. Recall that for a perfect conducting spherical surface one has $B_\perp = 0$ when $r = R$. And recall that $-\vec{\nabla}\phi_M = \mathbf{B}$, so:

$$\begin{aligned}
0 = -\partial_r|_{r=R} = -\partial_r &\left(-B_0\, r\cos\theta + \sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta)\right) = \\
-B_0\cos\theta &= \sum_{\ell=0}^{\infty}\frac{(\ell+1)R^\ell B_\ell}{R^{2\ell+2}}P_\ell(\cos\theta) = \\
B_1 &= -\frac{R^3 B_0}{2}.
\end{aligned} \tag{3.6.43}$$

[25] the field carried from the plane wave, but reflected.

So the first leading order of the "magnetic" potential are:

$$\begin{aligned}\phi_M(r,\theta) &= -B_0\, r\cos\theta\left(1+\tfrac{R^3}{2r^3}\right)\to\\ \phi_M(z) &= -B_0\, z\left(1+\tfrac{R^3}{2r^3}\right).\end{aligned} \tag{3.6.44}$$

Which will result into the following magnetic field:

$$\mathbf{B} = -\vec{\nabla}\phi_M = B_0\left(1-\tfrac{R^3}{2}\left(\tfrac{3\hat{r}\,z^2-1}{r^3}\right)\right). \tag{3.6.45}$$

2):

The absorption cross section is the ratio between the power loss and the intensity of the plane wave. Let's compute the power loss by:

$$\begin{aligned}\underbrace{\frac{P_{\text{loss}}}{da}}_{\text{area}} &= \frac{1}{2\sigma\delta}\left|\hat{n}\times\tfrac{\mathbf{B}}{\mu}\right|^2 =\\ &= \frac{1}{2\sigma\delta}\left|\hat{r}\times B_0\left(1-\tfrac{R^3}{2}\left(\tfrac{3\hat{r}\,z^2-1}{r^3}\right)\right)\right|^2 =\\ &= \frac{B_0^2}{2\mu^2}\left(1+\tfrac{R^3}{2r^3}\right)^2\sin^2\theta\,\hat{\phi}.\end{aligned} \tag{3.6.46}$$

So integrating this power loss over the whole area of the sphere we get:

$$\begin{aligned}P &= \int_{\text{Area}}\frac{B_0^2}{2\mu^2}\left(1+\tfrac{R^3}{2r^3}\right)^2\sin^2\theta R^2\sin\theta d\theta d\phi =\\ &= \frac{3B_0^2R^2\pi}{\sigma\delta\mu^2}.\end{aligned} \tag{3.6.47}$$

We only need to know the intensity of the incident plane wave. One can obtain this from the strength of both fields carried by the wave, which is given by the energy average of a wave, as:

$$\langle S \rangle = I = \frac{\epsilon_0}{2} |\mathbf{E}|^2 = \frac{1}{2\mu_0} |B_0|^2 \tag{3.6.48}$$

Then, the cross section is finally given by:

$$\sigma_{\text{cross}} = \frac{P_{\text{loss}}}{I} = \frac{6R^2\pi}{\sigma\delta\mu}. \tag{3.6.49}$$

Observe that expression (**??**) does not depend on the fields that the wave carries, but the properties of the object is scattered on. In this case, the radius, the conductivity and the skin depth..ubs.

3.6.7 Aperture (Science)

UNDER CONSTRUCTION

3.6.8 Born Scattering from a Dielectric Cube

1):

The first thing we have to notice in order to be able to approximate "a la Born" is that $\mathbf{E}$ inside a dielectric material has to be very similar to the incident electric field $\mathbf{E}$. If this then the case, we know that $\mathbf{J} \propto \mathbf{E}$. In order to simplify the fore coming expression, let us write $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ as the difference between the incoming and outgoing scattering wave vectors ($\omega = ck = ck_0$). Then, the approximated cross section is given by:

$$\frac{d\sigma_{\text{Born}}}{d\Omega} = \left(\frac{k_0^2 V \chi_e}{4\pi}\right)^2 \left|\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0\right|^2 \left|\int_V d^3r' \exp\left(i\mathbf{q} \cdot \mathbf{r}'\right)\right|^2. \tag{3.6.50}$$

Where χ is the susceptibility of the material of the cube. The pesky integral we have to compute goes as:

$$\int_V d^3r' \exp\left(i\mathbf{q} \cdot \mathbf{r}'\right) = \int_0^a dx' \exp\left(iq_x x'\right) \times \int_0^a dy' \exp\left(iq_y y'\right) \times \int_0^a dz' \exp\left(iq_z z'\right). \tag{3.6.51}$$

But we have to realise the following technical computation.

$$
\begin{aligned}
\int_0^a dx' \exp\left(iq_x x'\right) &= \frac{e^{iq_x x}}{iq_x}\Big|_0^a = \\
&= \underbrace{\frac{e^{iq_x a}}{iq_x} - 1}_{\text{extract } 1/2 \text{ of the exp}} \quad = \frac{e^{iq_x a/2}}{iq_x}(\underbrace{e^{iq_x a/2} - e^{-iq_x a/2}}_{=\sin\cdots}) = \\
&= 2e^{(-iq_x a/2)} \frac{\sin\left(q_x a/2\right)}{q_x a/2}
\end{aligned} \tag{3.6.52}
$$

Which works exactly the same for y, z components. Then, the cross section looks like:

$$
\frac{d\sigma_{\text{Born}}}{d\Omega} = \left(\frac{k_0^2 V \chi_e}{4\pi}\right)^2 \left|\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0\right|^2 \left[8 \frac{\sin\left(q_x a/2\right)}{q_x a/2} \frac{\sin\left(q_y a/2\right)}{q_y a/2} \frac{\sin\left(q_z a/2\right)}{q_z a/2}\right]^2 \tag{3.6.53}
$$

2):

Now we have to show that $\sigma \approx \frac{1}{4}k^2 a^4 \chi^2$ when $ka \gg 1$. Let $\mathrm{k}_0 = \hat{\mathrm{z}}$ and $\mathrm{E}_0 = \hat{\mathrm{x}}$. When $ka \gg 1$, near-forward scattering dominates and $\left|\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0\right| \approx \left|\hat{\mathbf{k}}_0 \times \hat{\mathbf{E}}_0\right| = 1$. The sketch below shows that, in the same limit, the area element $k^2\Omega_{\mathrm{k}}$ is essentially the area $dq_x dq_y$ of a circular disk perpendicular[26] to k_0:

$$
dS_{\mathrm{k}} = k^2 d\Omega_{\mathrm{k}} \approx dq_x dq_y. \tag{3.6.54}
$$

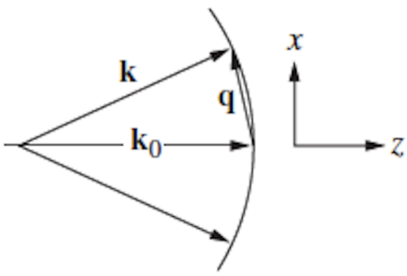


Figure 30: The important k to give an accurate approximation.

[26] Basically, try to imagine the to-be-scattered wave as a spherical one and the surface of this that will interact with the cube is given such that there is not too much difference of k in z direction.

Therefore, in the $ka \gg 1$ limit when $q_z \to 0$, the fact that $k_0 = k$ means that:

$$\lim_{ka\gg 1} \sigma_{\text{Born}} = \lim_{ka\gg 1} \int d\Omega_{\text{k}} \frac{d\sigma_{\text{Born}}}{d\Omega_{\text{k}}} \approx \lim_{ka\gg 1} \left(\frac{kV\chi_e}{4\pi}\right)^2 \int dq_x \frac{\sin^2\left(q_x a/2\right)}{\left(q_x a/2\right)^2} \int dq_y \frac{\sin^2\left(q_y a/2\right)}{\left(q_y a/2\right)^2}. \tag{3.6.55}$$

Here we have to be careful. The integrals are mainly dominated by contributions when $q_x, q_y \sim 1/a$ so the limits can be extended to $\pm\infty$[27] with little loss of accuracy. Therefore,

$$\lim_{ka\gg 1} \sigma_{\text{Born}} = \frac{k^2 a^4 \chi_e^2}{4\pi^2} \left[\int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\right]^2 \approx \frac{k^2 a^4 \chi_e^2}{4}. \tag{3.6.56}$$

3):

From the definition of the cross section and the result of part (1),

$$\frac{E_{\text{rad}}}{E_0} \approx \frac{1}{r}\sqrt{\frac{d\sigma}{d\Omega}} \approx \frac{k^2 a^3 \chi_e}{4\pi r} \left| \frac{\sin\left(q_x a/2\right)}{q_x a/2} \frac{\sin\left(q_y a/2\right)}{q_y a/2} \frac{\sin\left(q_z a/2\right)}{q_z a/2} \right|. \tag{3.6.57}$$

The absolute-value term gets no larger than one. Therefore, with $r = a$, the weak scattering criterion is indeed

$$1 \gg k^2 a^2 \chi_e = \sigma_{\text{Born}}/a^2 \chi_e. \tag{3.6.58}$$

3.6.9 E: Two Antennas Sitting Together

1):

As almost all radiation problems, let's start computing the different dipole momenta for each of the components of the system. Recall that when integrating we only care about the spatial components. The magnetic dipole for the loop antenna is:

$$\vec{m}_{\text{loop}} = \tfrac{1}{2} \int \vec{x} \times \vec{J}(\vec{x}) d^3x = I \times \text{Area}_{S_2} \cdot \hat{n} = I_0 \pi a^2 \hat{z}. \tag{3.6.59}$$

[27] The peak of contribution happens at $1/a$ for the momenta with the remaing values of q_i barely contributing to the situation. Expanding the integration to all R we simplify our life.

Where we shall recall that for a close loop we can use the previous magnetic dipole simplification. Also, the dipole will be pointing perpendicular to the surface of the area. For the electric dipole of the straight antenna we find:

$$\vec{P}_{\text{ant}} = \int \vec{x}\cdot\rho(\vec{x})d^3x = \int_{-a}^{0} -i\lambda_0\, z\, dz + \int_0^a i\lambda_0\, z\, dz = i\lambda_0\, a^2\,\hat{z}. \tag{3.6.60}$$

2):

Assuming that the emitted wavelength is greater than the size of the system and that we are far far away of the source, we can just drop terms with a $\frac{1}{r^n}$ dependence in the vector potential expression **A**. Also, as we have two antennas, whose dipoles are aligned along the z-axis, the total vector potential will be no more than a linear combination of each of the parts as:

$$\begin{aligned}\mathbf{A}_{total} &= \mathbf{A}_{ant}(\vec{x}) + \mathbf{A}_{loop}(\vec{x}) = \\ &= \frac{ik\mu_0}{4\pi}(\hat{r}\times\vec{m})\frac{e^{ikr}}{r} - \frac{i\mu_0\omega}{4\pi}\vec{p}\frac{e^{ikr}}{r} = \\ &= \frac{i\mu_0\omega e^{ikr}}{4\pi\, r}\left(\tfrac{1}{c}\pi I_0 a^2\hat{r}\times\vec{z} - i\lambda_0 a^2\hat{z}\right).\end{aligned} \tag{3.6.61}$$

We will leave this expression for future convenience in the computation.

3):

Same principle as before, **E** and **B** will have contributions from both antennas. In this case, let's start with:

$$\begin{aligned}\mathbf{H}_{tot} &= \mathbf{H}_{ant} + \mathbf{H}_{loop} = \\ &= \frac{e^{ikr}k^2}{4\pi r}\left(c(\hat{r}\times\vec{p}) - (\hat{r}\times\vec{m})\times\hat{r}\right) = \text{use vector identities} = \\ &= \frac{e^{ikr}k^2}{4\pi r}\left(c(\hat{r}\times\hat{z})\, i\lambda_0 a^2 - \pi I_0 a^2\hat{z} - \pi I_0 a^2 \underbrace{\cos\theta\,\hat{r}}_{\vec{m}\cdot\hat{r}=\cos\theta}\right).\end{aligned} \tag{3.6.62}$$

Then it is easy to consider that $\mathbf{B} = \mu_0\mathbf{H}_{tot}$. For the electric field, as we are in the radiation zone, we know that:

$$
\begin{aligned}
\mathbf{E}_{tot} &= Z_0 \mathbf{H}_{tot} \times \hat{r} = \\
&= \sqrt{\tfrac{\mu_0}{\epsilon_0}} k^2 \frac{e^{ikr}}{4\pi r} \left(c\, i\lambda_0 a^2 (\hat{r} \times \vec{z}) \times \hat{r} + \pi I_0 a^2 \hat{r} \times \hat{z} - \cancel{\hat{r} \times \hat{r}} \right) = \\
&= \text{Again, use some vector identities to simplify...} = \\
&= \frac{-k^2}{4\pi r} \sqrt{\tfrac{\mu_0}{\epsilon_0}} \frac{e^{ikr}}{r} (i\, c\lambda_0 a^2 (\hat{z} - \hat{r})).
\end{aligned}
\tag{3.6.63}
$$

Then, finally, we are prepared to compute the power emitted per unit of solid angle...

4):

So we know it expression as:

$$
\begin{aligned}
\left\langle \frac{dP}{d\Omega} \right\rangle &= \frac{1}{2} \Re \left[r^2 \hat{r} \cdot (\vec{E} \times \vec{H}^*) \right] = \\
&= \frac{1}{2} \Re \left[r^2 \hat{r} \cdot \left(\sqrt{\tfrac{\mu_0}{\epsilon_0}} \left(\vec{H} \times \hat{r} \right) \times \vec{H}^* \right) \right] = \\
&= \text{use vector identities} = \\
&= \frac{1}{2} \Re \left[r^2 \sqrt{\tfrac{\mu_0}{\epsilon_0}} \vec{H} \cdot \vec{H}^*) \right].
\end{aligned}
\tag{3.6.64}
$$

We can suspect what is about to come. One has to compute the conjugated square of expression (**??**). Grab a coffee, keep calm and compute with ease and patience. A result of the following form should pop up as:

$$
\left\langle \frac{dP}{d\Omega} \right\rangle = \sqrt{\tfrac{\mu_0}{\epsilon_0}} \frac{k^4}{16\pi^2} \left(c^2 \lambda_0^2 a^4 \underbrace{(\vec{r} \times \hat{z})^2}_{1-(\vec{r}\cdot\hat{z})^2} + 2\pi^2 I_0^2 a^4 (1 + \underbrace{\hat{z} \cdot \vec{r}}_{-\cos\theta}) \right). \tag{3.6.65}
$$

We are asked to give the answer as a function of the angle θ. In order to simplify further the previous expression, recall that $\cos\theta = 1 - 2\sin^2\theta/2$. With this in mind, the final result looks like:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{k^4 a^4}{32\pi^2} \left(c^2 \lambda_0^2 \sin^2\theta + 4\pi^2 I_0^2 \sin^2\frac{\theta}{2} \right). \tag{3.6.66}$$

3.6.10 E: One... Err, Two Antennas

UNDER CONSTRUCTION

3.6.11 E : Who bent my Antenna?

1):

This first part of the problem is just a warm up for what is about to come. Let's start computing the magnetic dipole. Careful here; We cannot assume a whole curved antenna, as there is a gap at π, that we shall not integrate over. It is useful to move to cylindrical coordinates when computing in this exercise. Then we have:

$$\begin{aligned} \vec{m} &= \tfrac{1}{2} \int d^3x\, \vec{x} \times \vec{J} = \tfrac{1}{2} \int d^3x\, (r, \alpha, 0) \times (0, I, 0) = \\ &= I_0 a^2 \hat{x} \int_{\pi}^{-\pi} d\alpha (\pi - |\alpha|) = \pi^2 I_0 a^2 \hat{x}. \end{aligned} \tag{3.6.67}$$

To find the electric dipole we need first to know the charge density. This one can be extracted from the continuity equation as:

$$\vec{\nabla} \cdot \vec{J} + \partial_t \rho = 0 \rightarrow \lambda = \tfrac{-i}{a\omega} \partial_\alpha (I_0(\pi - |\alpha|)) \tag{3.6.68}$$

Which depends on the value of α (Recall that it is fed with a RF signal at $\alpha = 0$). This means:

$$\lambda = \begin{cases} \frac{i}{a\omega} I_0 & 0 < \alpha < \pi, \\ \frac{-i}{a\omega} I_0 & -\pi < x < 0. \end{cases} \tag{3.6.69}$$

By symmetry $\vec{p}$ is in the $\hat{z}$ direction as:

$$\vec{p} = \int d^3x\, \lambda\, \hat{z} = \tfrac{2i}{\omega} I_0 a \hat{z} \int_0^{\pi} d\alpha \sin\alpha = \tfrac{4i}{\omega} I_0 a \hat{z}. \tag{3.6.70}$$

2):

At this point we could compute each of the contributions for each dipole to the vector field... Or we can be intelligent and save some time and brain cells by doing the following. We know that the vector potential is:

$$\begin{aligned}
\mathbf{A}(\vec{x})_p &= -\frac{i\mu_0 \overbrace{\omega}^{c=k\omega}}{4\pi}\vec{p}\frac{e^{ikr}}{r}, \\
\mathbf{A}(\vec{x})_m &= \frac{ik\mu_0}{4\pi}\vec{r}\times\vec{m}\left(1-\frac{1}{ikr}\right)\frac{e^{ikr}}{r}.
\end{aligned} \tag{3.6.71}$$

So the sum of both of them in the radiation zone (a.k.a big r so drop terms $\propto \frac{1}{r}$) will yield the following result:

$$\mathbf{A}(\vec{x}) = \frac{ik\mu_0}{4\pi}\left(\vec{r}\times\vec{m}-c\vec{p}\right)\frac{e^{ikr}}{r} = \frac{i\mu_0}{4\pi} I_0 ka\left(\pi^2 ka(\hat{r}\times\hat{x}) - 4i\hat{z}\right)\frac{e^{ikr}}{r}. \tag{3.6.72}$$

Observe that we have left the vector products as a formal expression. This has been done so we can exploit its identities later. With this expression in our power, is time to compute the fields.

3):

The fields are given by:

$$\begin{aligned}
\mathbf{B} &= \vec{\nabla}\times\vec{A} =, \\
&= \frac{\mu_0}{4\pi} I_0 ka\left(\pi^2 ka\hat{r}\times(\hat{r}\times\hat{x}) + 4i\hat{r}\times\hat{z}\right)\frac{e^{ikr}}{r}.
\end{aligned} \tag{3.6.73}$$

And the Electric field is (in the radiation zone):

$$\begin{aligned}
\mathbf{E} &= c^2\tfrac{i}{\omega}\vec{\nabla}\times\vec{B} =, \\
&= \frac{c\mu_0}{4\pi} I_0 ka\left(\pi^2 ka(\hat{x}\times\hat{r}) + 4i\hat{r}\times(\hat{z}\times\hat{r})\right)\frac{e^{ikr}}{r}.
\end{aligned} \tag{3.6.74}$$

4):

Finally, we have all the tools to compute the power emitted by solid angle. Recall that this is given by:

$$\frac{dP}{d\Omega} = \frac{r^2}{2\mu_0}\hat{r}\cdot\Re\left[\mathbf{E}\times\mathbf{B}^*\right] = \text{radiation zone} = \frac{r^2}{2\mu_0 c}\left(\vec{E}\cdot\vec{E}^*\right). \tag{3.6.75}$$

This forces us to carefully compute the set of vector products in the electric field. With some calm (and vector identities) we can find that:

$$\begin{aligned} |\left(\pi^2 ka(\hat{x}\times\hat{r}) + 4i\hat{r}\times(\hat{z}\times\hat{r})\right)|^2 &= \pi^4k^2a^2(\hat{x}\times\hat{r})(\hat{x}\times\hat{r}) + 16(\hat{z}-\hat{r}(\hat{r}\cdot\hat{z}))^2 = \\ &= \pi^4k^2a^2(1-(\hat{x}\cdot\hat{r})^2) + 16(1-(\hat{r}\cdot\hat{z})^2) =, \\ &= \pi^4k^2a^2(1-(\sin\theta\cos\phi)^2) + 16(1-(\cos\theta)^2). \end{aligned} \tag{3.6.76}$$

Plugging this back into expression (**??**) we find:

$$\frac{dP}{d\Omega} = \frac{c\mu_0}{32\pi^2}(I_0ka)^2\left(\pi^4k^2a^2(1-(\sin\theta\cos\phi)^2) + 16\sin\theta^2)\right). \tag{3.6.77}$$

3.7 Covariant Formalism of Electrodynamics

3.7.1 Getting familiar with four-vectors

1)

What is then $\partial_\mu x^\mu$? The answer is given in the statement itself; A scalar. A function. In this course we are going to indices up (x^μ) and down (x_μ) to denote the components of objects like vectors, forms and tensors. Whenever we see two indices of the same form repited up and down, this means that we are using the Einstein's summation convenction. This means that:

$$\partial_\mu x^\mu = \sum_{\mu=0}^{3}\frac{\partial}{\partial x^\mu}x^\mu = \frac{\partial}{\partial x^0}x^0 + \frac{\partial}{\partial x^1}x^1 + \frac{\partial}{\partial x^2}x^2 + \frac{\partial}{\partial x^3}x^3. \tag{3.7.1}$$

To see that this is a Lorentz scalar, we just need to check that it transforms as one. In this case we just need to know how each part of this object transforms. We know that forms Lorentz transform as:

$$\partial_\mu \quad \rightarrow \quad \frac{\partial x^\alpha}{\partial x^\mu \partial x^\alpha}. \tag{3.7.2}$$

And the vector entries x^μ do as:

$$x^\mu \quad \to \quad \frac{\partial x^\mu}{\partial x^\alpha} x^\alpha \tag{3.7.3}$$

So, the whole object (**??**) transforms as:

$$\partial_\mu x^\mu \quad \to \quad \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\alpha} \frac{\partial x^\mu}{\partial x^\alpha} x^\alpha = \partial_\alpha x^\alpha. \tag{3.7.4}$$

As the repeated indices are summing, we can rewrite them again to be μ. When two indices are summing in this way, it is said that they are dummy indices, so they can be replaced by any other letter. As we see, the scalar transforms according to Lorentzian transformations.

2)

To understand this we just have to check the dimension of the objects. A $F^{\mu\nu}$ (two-tensor) has $D \times D$ entries[28]. But in the case of something of the form $\sum_{\mu\nu} a^\mu b^\nu$ we just have a dimension of $2D$ entries.

3)

a) In this section we have to abuse of the (anti)symmetry of the given tensors. There is nothing that does not allow us to do the following:

$$\begin{aligned} S_{\mu\nu} &= \tfrac{1}{2} S_{\mu\nu} + \tfrac{1}{2} S_{\mu\nu}, \\ &= \tfrac{1}{2} S_{\mu\nu} + \tfrac{1}{2} \underbrace{S_{\nu\mu}}_{\text{symmetric}} . \end{aligned} \tag{3.7.5}$$

If we apply a Lorentz transformation on both sides of the equation we find:

$$\begin{aligned} \Lambda^\mu_\alpha \Lambda^\nu_\beta S_{\mu\nu} &= \tfrac{1}{2} \Lambda^\mu_\alpha \Lambda^\nu_\beta S_{\mu\nu} + \tfrac{1}{2} \underbrace{\Lambda^\nu_\alpha \Lambda^\mu_\beta S_{\nu\mu}}_{\mu,\nu \text{ dummy Change.}} , \\ &= \tfrac{1}{2} \Lambda^\mu_\alpha \Lambda^\nu_\beta S_{\mu\nu} + \tfrac{1}{2} \Lambda^\mu_\alpha \Lambda^\nu_\beta S_{\mu\nu}, \\ &= \Lambda^\mu_\alpha \Lambda^\nu_\beta S_{\mu\nu}. \end{aligned} \tag{3.7.6}$$

[28]This is a regular property of the tensor product notation.

It can be similarly done for the antisymmetric case.

b)

We want to prove that the product of an antisymmetric and a symmetric tensors is 0. We could define generic ones and compute term by term to see that they cancel... or we can apply so facts, so we can reduce the amount of work. Let's aim for the second option. We can exploit the (anti)symmetry of the tensors as:

$$S^{\mu\nu}A_{\mu\nu} \quad \rightarrow \quad -S^{\nu\mu}A_{\nu\mu}. \tag{3.7.7}$$

But we know that when indices are repited, that means that they are dummy... So we can replace them on convenience. Let's interchange then $\mu \leftrightarrow \nu$, so.

$$S^{\mu\nu}A_{\mu\nu} = -S^{\mu\nu}A_{\mu\nu}. \tag{3.7.8}$$

As everything is contracted (a.k.a all the indices are summed) we know that the result is a scalar. Which is the only scalar that equal to its opposite? Zero.

3.7.2 Covariant formalism of Electrodynamics

a)

In order to solve this problem, we first require the explicit expression for the electromagnetic field tensor. This is given by:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.7.9}$$

Observe that the expression (**??**) is given in S.I. units. In this case, we can relate each of the components of the tensor with its entries such that:

$$F^{0i} = -E^i, \quad F^{ij} = \epsilon^{ijk}B_k, \quad F^{\mu\nu} = -F^{\nu\mu}. \tag{3.7.10}$$

As we want to get used to this notation (which can be new to some of you) let explicitly write all terms when summing over the Einstein's convention for the sumation. In this case it goes as:

$$
\begin{aligned}
-F_{\mu\nu}F^{\mu\nu} = &-\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} F_{\mu\nu}F^{\mu\nu} = \\
= &-(F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + \\
&+ F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13} + \\
&+ F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} + \\
&+ F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33}).
\end{aligned}
\tag{3.7.11}
$$

So far, we know the aspect of the entries $F^{\mu\nu}$, but we do not know how the entries of the tensor with two indices down ($F_{\mu\nu}$) look like. We can compute them using the metric of this space. In this case, we are working on a Minkowski space, so recall then:

$$
\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.7.12}
$$

So "lowering" the indices will be a process of the form:

$$
F_{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}F^{\mu\nu}. \tag{3.7.13}
$$

One can also reduce the problem to a matrix multiplication problem, making use of the fact that $F' = \eta^T F \eta$, where η^T is the transpose metric matrix. This is not recommendable, as in the future we will be working with objects with more indices which does not have a matrix representation". The index formalism is powerful enough with any tensor that will appear during this course. If one performs then the calculation in (**??**), it can be found that:

$$
-F_{\mu\nu}F^{\mu\nu} = 2(\vec{E}^2 - \vec{B}^2). \tag{3.7.14}
$$

One could perform the same computation for $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F_{\rho\sigma}$. Or we can make use of the properties of the dual tensor (Eq 11.140 Jackson). The dual tensor "interchanges" the position of the entries E^i and B^i such that:

$$
E^i \to -B^i, \quad B^i \to E^i. \tag{3.7.15}
$$

In this case, we can interchange the electric field and magnetic field components appearing in **??** as:

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F_{\rho\sigma} = *F^{\rho\sigma}F_{\rho\sigma} = -\vec{E}\cdot\vec{B}. \tag{3.7.16}$$

b)

How can we show that Maxwell equations are identical to the Bianchi identity? As we are relating two equations to one identity, this means that we can extract two possible forms of the identity that will have a similar appearance to Maxwell's. It will be useful to use relations (**??**) to "massage" the appearance of the entries.
Let's start the second Maxwell's. It can be written as:

$$\vec{\nabla}\cdot\vec{B} = \partial_i B^i = \frac{\partial}{\partial x^1}B^1 + \frac{\partial}{\partial x^2}B^2 + \frac{\partial}{\partial x^3}B^3. \tag{3.7.17}$$

Here we will use relations (**??**) to express B in terms of F^{ij} entries. This means that B^i can be express as:

$$F_{ij} = \epsilon_{ijk}B^k \quad \rightarrow \quad \epsilon^{ijk}F_{ij} = \underbrace{\epsilon^{ijk}\epsilon_{ijk}}_{1}B^k \quad \rightarrow \quad \epsilon^{ijk}F_{ij} = B^k. \tag{3.7.18}$$

So the second Maxwell's can be written as:

$$\partial_i B^i = \partial_i(\epsilon^{kji}F_{kj}) = \partial_i(-\epsilon^{ijk}(-F_{jk})) = \epsilon^{ijk}\partial_i(F_{jk}) = 0. \tag{3.7.19}$$

Where $\epsilon^{ijk}\partial_i(F_{jk})$ represent the non repeating permutations of each of the entries for i, j, k.

To prove the equivalence of the first Maxwell's equation to the Bianchi identity, assume that t is the zero-th entry in Bianchi such that:

$$\partial_0 F_{\nu\lambda} + \partial_\nu F_{\lambda 0} + \partial_\lambda F_{0\nu} = 0. \tag{3.7.20}$$

But we know that $F^{0i} = E^i$. Then, using antisymmetry of the electromagnetic tensor we have:

$$\partial_0 F_{\nu\lambda} \qquad \underbrace{-\partial_\nu E_\lambda + \partial_\lambda E_\nu}_{\text{j-th compo. of cross product}} \qquad = 0. \tag{3.7.21}$$

So we just have to massage what $F_{\nu\lambda}$ is. We know from relations(**??**) that it corresponds to $F_{ij} = -\epsilon_{ijk}B^k$. So we have:

$$\begin{gathered}\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = \partial_0 F_{\nu\lambda} + \partial_\nu F_{\lambda 0} + \partial_\lambda F_{0\nu} = \\ \partial_0(-\epsilon_{k\nu\lambda}B^k) - \partial_\nu E_\lambda + \partial_\lambda E_\nu = \partial_0(-\epsilon_{k\nu\lambda}B^k) - (\vec{\nabla}\times\vec{E})^k = \\ \partial_t B^k + (\vec{\nabla}\times\vec{E})^k = \partial_t\vec{B} + (\vec{\nabla}\times\vec{E}) = 0.\end{gathered} \tag{3.7.22}$$

c)

We are given $T^{\mu\nu} = F^\mu_\rho + F^{\rho\nu} - \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}$. This means that from the left hand side (LHS) we can compute 3 different options: T^{00}, T^{0i}, T^{ij}.

- T^{00}

$$\begin{aligned}T^{00} = F^0_\rho F^{\rho 0} - \frac{1}{4}g^{00}(F_{\rho\sigma}F^{\rho\sigma}) = -g^{00}F_{0\rho}F^{0\rho} - \frac{1}{4}(-1)(F_{\rho\sigma}F^{\rho\sigma}) = \\ = \vec{E}^2 - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) = \frac{1}{2}(\vec{E}^2 + \vec{B}^2).\end{aligned} \tag{3.7.23}$$

- T^{0i}

$$\begin{aligned}T^{0i} = F^0_\rho F^{\rho i} + \frac{1}{2}\underbrace{g^{0i}}_{\text{by symmetry is 0.}}\cdots = \\ = g^{00}F_{0\rho}F^{\rho i} = -(E_jB_k - E_kB_j) = (\vec{E}\times\vec{B})^i = S^i_{\text{poynt}}\end{aligned} \tag{3.7.24}$$

- T^{ij}:

$$\begin{aligned}T^{ij} &= -g^{ij}F_{i\rho}F^{j\rho} - \frac{1}{4}g^{ii}(-2(\vec{E}^2 - \vec{B}^2)) = \\ &= -g^{ij}F_{i0}F^{j0} - g^{ij}F_{ik}F^{jk} + \frac{1}{2}g^{ij}((\vec{E}^2 - \vec{B}^2)) = \\ &= -E^iE^j + (B^iB^j) - \frac{1}{2}(\vec{E}^2 - \vec{B}^2)\delta^{ij}.\end{aligned} \tag{3.7.25}$$

d)

To show that ϵ is invariant under Lorentz transformations, we have to consider that it is not a tensor, but a pseudo-tensor. When transofrmed, it should be multiplied by the inverse of the Jacobian of the transformation; In our case, the transformation has to do with the metric $g_{\mu\nu}$, so $\det g = -1$. This metric is Lorentzian, so no problem.

There is a nice property of tensors that goes as:

$$\epsilon_{\mu\ldots\zeta} A^{\mu}_{a} \ldots A^{\zeta}_{z} = \det A \, \epsilon_{a\ldots z}. \tag{3.7.26}$$

We can apply this to our case such that:

$$\epsilon^{0123} = \Lambda^0_\alpha \Lambda^1_\beta \Lambda^2_\gamma \Lambda^3_\delta \epsilon^{\alpha\beta\gamma\delta} = \underbrace{\det \Lambda}_{\text{It is 1}} \epsilon^{\alpha\beta\gamma\delta}. \tag{3.7.27}$$

As the determinant of Lorentzian transformations is equal to 1, we have just proved it.

3.7.3 Lorentz Transformations for the Electromagnetic Field

a) We are asked to show the general Lorentz transformation of the electric and magnetic fields. As a warm-up for this problem, it is recommendable to check first a simpler case; Let's assume we only move along the x axis. The Lorentz transformation of the electromagnetic tensor (which includes $\vec{E}$ and $\vec{B}$) can be thought as a matrix calculation of the form:

$$F' = \Lambda F \Lambda^t, \tag{3.7.28}$$

Where Λ is the boost matrix in the x direction. Then:

$$F' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_\parallel & -E_\perp & -E_\perp \\ E_\parallel & 0 & -B_\perp & B_\perp \\ -E_\perp & B_\perp & 0 & -B_\parallel \\ -B_\perp & -B_\perp & B_\parallel & 0 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \tag{3.7.29}$$

$$= \begin{pmatrix} 0 & -E_\parallel & -\gamma(E_\perp - \beta B_\perp) & -\gamma(E_\perp + \beta B_\perp) \\ E_\parallel & 0 & \gamma(\beta E_\perp - B_\perp) & \gamma(\beta E_\perp + B_\perp) \\ \gamma(E_\perp - \beta B_\perp) & -\gamma(\beta E_\perp - B_\perp) & 0 & -B_\parallel \\ \gamma(E_\perp + \beta B_\perp) & -\gamma(\beta E_\perp + B_\perp) & B_\parallel & 0 \end{pmatrix}$$

Where we have used the fact that $1 - \beta^2 = \gamma^{-2}$. One can see then that:

$$
\begin{aligned}
E'_{\parallel} &= E_{\perp}, \qquad E'_{\perp} = \gamma(E_{\perp} + \beta \times B_{\perp}),\\
B'_{\parallel} &= B_{\perp}, \qquad B'_{\perp} = \gamma(B_{\perp} - \beta \times E_{\perp}).
\end{aligned} \tag{3.7.30}
$$

But this was for a specific case. How does a general Lorentz transformation looks like?

To see how a general transformation would look like, we need the more general expresion of the boost matrix. Λ. This is:

$$
\Lambda^{\mu}_{\nu} = \begin{pmatrix}
-\gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\
-\gamma\beta_x & 1+(\gamma-1)\frac{v_x v_x}{v^2} & (\gamma-1)\frac{v_x v_y}{v^2} & (\gamma-1)\frac{v_x v_x}{v^2} \\
-\gamma\beta_y & (\gamma-1)\frac{v_x v_y}{v^2}) & 1+(\gamma-1)\frac{v_y v_y}{v^2} & (\gamma-1)\frac{v_y v_z}{v^2} \\
-\gamma\beta_z & (\gamma-1)\frac{v_x v_z}{v^2} & (\gamma-1)\frac{v_z v_y}{v^2} & 1+(\gamma-1)\frac{v_z v_z}{v^2}
\end{pmatrix}. \tag{3.7.31}
$$

So, as life is short and we want to wisely use our time, let's use the indices notation to be more productive with our time. Recall:

$$
F^{\mu\nu'} = \Lambda^{\mu}_{å}\Lambda^{\nu}_{\beta}F^{å\beta}. \tag{3.7.32}
$$

So the only thing that we have to do is to calculate each of the entries of $F^{\mu\nu}$. This goes as:

- $F^{00'}$

$$
\begin{aligned}
F^{00'} &= \Lambda^0_0\Lambda^0_\beta F^{0\beta} + \Lambda^0_1\Lambda^0_\beta F^{1\beta} + \Lambda^0_2\Lambda^0_\beta F^{2\beta} + \Lambda^0_3\Lambda^0_\beta F^{3\beta},\\
&= \Lambda^0_0\Lambda^0_0 F^{00} + \cdots + \Lambda^0_1\Lambda^0_0 F^{10} + \cdots + \Lambda^0_2\Lambda^0_0 F^{20} + \cdots + \Lambda^0_3\Lambda^0_0 F^{30},\\
&= \text{Observe that } \Lambda \text{ commutes with each other and } F^{ij} \text{ is antisymmetric}\\
&= 0.
\end{aligned} \tag{3.7.33}
$$

- $F^{0i'}$

$$
\begin{aligned}
F^{0i'} =& \Lambda^0_0\Lambda^i_0 F^{00} + \cdots + \Lambda^0_1\Lambda^i_0 F^{10} + \cdots +\\
&+ \Lambda^0_2\Lambda^i_0 F^{20} + \cdots + \Lambda^0_3\Lambda^i_0 F^{30} + \cdots =\\
=& (\gamma\Lambda^i_1 + \gamma\beta_x\Lambda^i_0)(-E_x) + \cdots + (\gamma\Lambda^i_3 + \gamma\beta_z\Lambda^i_0)(-E_z) +\\
&+ (-\gamma\beta_y\Lambda^i_3 + \gamma\beta_z\Lambda^i_2)(B_x) + \cdots + (-\gamma\beta_x\Lambda^i_2 + \gamma\beta_y\Lambda^i_1)(B_z).
\end{aligned} \tag{3.7.34}
$$

Terms of the form F^{ij} are left as an exercise for the reader[29]. One can see in eq(**??**) that the presence of the magnetic field entries corresponds to the cross products that are present in expression (**??**). So a general boost will have the same terms as in that expression.

b)

In order to argue what happens to the angle between $\vec{E}$ and $\vec{B}$, recall that θ is given by:

$$\vec{E}\cdot\vec{B} = |\vec{E}||\vec{B}|\cos\theta. \tag{3.7.35}$$

So the cosine will behave like:

$$\cos\theta = \frac{\vec{E}\cdot\vec{B}}{|\vec{E}||\vec{B}|} = \frac{E_\parallel B_\parallel + E_\perp B_\perp}{\sqrt{E_\parallel^2+E_\perp^2}\sqrt{B_\parallel^2+B_\perp^2}}. \tag{3.7.36}$$

Which after boosting looks:

$$\cos\theta' = \frac{\vec{E}'\cdot\vec{B}'}{|\vec{E}'||\vec{B}'|} = \frac{E_\parallel B_\parallel + \gamma^2(E_\perp+\beta\times B_\perp)(B_\perp-\beta\times E_\perp)}{\sqrt{E_\parallel^2+\gamma^2(E_\perp+\beta\times B_\perp)^2}\sqrt{B_\parallel^2+\gamma^2(B_\perp-\beta\times E_\perp)^2}}. \tag{3.7.37}$$

It can be easily seen from the previous expression that θ between $\vec{E}'$ and $\vec{B}'$ will change respect to its previous configuration. This change will depend on the velocity on which the system moves respect to a specific frame.

3.7.4 Three Observers. "One Field"

1):

Observer A evaluates the two electromagnetic field invariants and finds the values

$$\mathbf{E}\cdot\mathbf{B} = \alpha^2/c, \qquad E^2 - c^2B^2 = \alpha^2 - c^2\left(\alpha^2/c^2 + 4\alpha^2/c^2\right) = -4\alpha^2. \tag{3.7.38}$$

Observer B evaluates the same invariants and finds $\mathbf{E}'\cdot\mathbf{B}' = E'_x\alpha/c + B'_y\alpha$ and

[29]This is more common to be found in a mathematics textbook. Surprise! I was not in the mood of typing that entry...

$$
\begin{aligned}
E'^2 - c^2 B'^2 &= E_x'^2 + \alpha^2 - c^2\left(2\alpha^2/c^2 + B_y'^2\right), \\
&= E_x'^2 - c^2 B_y'^2 - \alpha^2.
\end{aligned}
\tag{3.7.39}
$$

Setting these invariants equal in the two frames gives:

$$
\begin{aligned}
E_x' + cB_y' &= \alpha, \\
E_x'^2 - c^2 B_y'^2 &= -3\alpha^2.
\end{aligned}
\tag{3.7.40}
$$

Which after solving, we find $E_x' = -\alpha$ and $B_y' = 2\alpha/c$. Therefore:

$$
\mathbf{E}' = (-\alpha, \alpha, 0), \qquad \mathbf{B}' = (\alpha/c, 2\alpha/c, \alpha/c). \tag{3.7.41}
$$

2):

The fields transform according to:

$$
\begin{aligned}
\mathbf{E}_\parallel' &= \mathbf{E}_\parallel, \qquad \mathbf{E}_\perp' = \gamma(\mathbf{E} + \boldsymbol{\beta}\times c\mathbf{B})_\perp \\
\mathbf{B}_\parallel' &= \mathbf{B}_\parallel, \qquad c\mathbf{B}_\perp' = \gamma(c\mathbf{B} - \boldsymbol{\beta}\times \mathbf{E})_\perp
\end{aligned}
\tag{3.7.42}
$$

Therefore,

$$
\begin{aligned}
\mathbf{E}'' &= -\alpha\hat{\mathbf{x}} + \gamma\left[\mathbf{E}_\perp' + v\hat{\mathbf{x}}\times\left(B_y'\hat{\mathbf{y}} + B_z'\hat{\mathbf{z}}\right)\right] = \\
&= -\alpha\hat{\mathbf{x}} + \gamma\left[\mathbf{E}_\perp' + vB_y'\hat{\mathbf{z}} - vB_z'\hat{\mathbf{y}}\right] = \\
&= -\alpha\hat{\mathbf{x}} + \gamma(\alpha - v\alpha/c)\hat{\mathbf{y}} + 2\gamma v\alpha/c\hat{\mathbf{z}} = \\
&= -\alpha\hat{\mathbf{x}} + \gamma\alpha(1-\beta)\hat{\mathbf{y}} + 2\gamma\alpha\beta\hat{\mathbf{z}}.
\end{aligned}
\tag{3.7.43}
$$

Similarly we find for $\mathbf{B}$,

$$
\begin{aligned}
\mathbf{B}'' &= \alpha/c\hat{\mathbf{x}} + \gamma\left[\mathbf{B}_\perp' - \left(v/c^2\right)\hat{\mathbf{x}}\times\left(E_y'\hat{\mathbf{y}} + E_z'\hat{\mathbf{z}}\right)\right] = \\
&= \alpha/c\hat{\mathbf{x}} + \gamma\left[\mathbf{B}_\perp' - \left(v/c^2\right)E_y'\hat{\mathbf{z}} + \left(v/c^2\right)E_z'\hat{\mathbf{y}}\right] = \\
&= \alpha/c\hat{\mathbf{x}} + 2\gamma\alpha/c\hat{\mathbf{y}} + \gamma\left(\alpha/c - v\alpha/c^2\right)\hat{\mathbf{z}} = \\
&= \alpha/c\hat{\mathbf{x}} + 2\gamma\alpha/c\hat{\mathbf{y}} + \gamma\alpha(1-\beta)/c\hat{\mathbf{z}}.
\end{aligned}
\tag{3.7.44}
$$

3.7.5 Transformation of Force

a)

From Gauss law, the electric field inside the electron column is given by:

$$\mathbf{E} = \frac{\rho_0 r}{2\epsilon_0}\hat{\mathbf{r}} \quad r < a. \tag{3.7.45}$$

Therefore the force on a electron at $r < a$ is

$$\mathbf{F} = -eq\mathbf{E}(r) = -\frac{e\rho_0 r}{2\epsilon_0}\hat{\mathbf{r}}. \tag{3.7.46}$$

b)

In the laboratory frame of the observer,

$$\begin{aligned}
\mathbf{E}_\perp &= \gamma\left(\mathbf{E}' - \mathbf{v}\times\mathbf{B}'\right)_\perp, \\
\mathbf{E}'_\parallel &= \mathbf{E}_\parallel, \\
\mathbf{B}_\perp &= \gamma\left(\mathbf{B}' + \left(\mathbf{v}/c^2\right)\times\mathbf{E}'\right)_\perp, \\
\mathbf{B}'_\parallel &= \mathbf{B}_\parallel.
\end{aligned} \tag{3.7.47}$$

There is no magnetic field in the rest frame of the electrons and the rest-frame electric field (computed in previous part of the exercise) is entirely transverse. Therefore, the force we are looking for is:

$$\mathbf{F} = -e\mathbf{E} = -e[\mathbf{E} + \mathbf{v}\times\mathbf{B}] = -e\gamma\mathbf{E}'_\perp - e\mathbf{v}\times\gamma\left[\left(\mathbf{v}/c^2\right)\times\mathbf{E}'_\perp\right] = -\frac{e\mathbf{E}'_\perp}{\gamma}. \tag{3.7.48}$$

3.7.6 A Long Wire Moving Fast

a)

The first part of this calculation is something easy from elementary electromagnetism. We are in the rest frame K'. For simplicity, assume that the wire is moving along z direction. In this case we are moving in the same reference system as the wire is moving, so we will not see any charge with velocity different than 0. As there are no moving charges, $\vec{B}' = 0$. For the electric field we have:

$$\int_{\text{cylinder}} \vec{E}' \cdot \vec{ds} = \tfrac{q}{\epsilon_0} \quad \rightarrow \quad \vec{E}' = \tfrac{q}{2\pi r L} \hat{r}. \tag{3.7.49}$$

And now, we use the Lorentz transformation of the fields to move them to the laboratory frame. Recall that they look like:

$$\begin{aligned} \vec{E} &= \gamma(\vec{E}' + \beta \times \vec{B}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{E}'), \\ \vec{B} &= \gamma(\vec{B}' - \beta \times \vec{E}') - \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{B}'). \end{aligned} \tag{3.7.50}$$

So:

$$\begin{aligned} \vec{E} &= \gamma\vec{E}' = \gamma\tfrac{q}{2\pi r L}\hat{r}, \\ \vec{B} &= -\gamma\vec{\beta} \times \vec{E}' = -\gamma\tfrac{q\beta}{2\pi r L}\hat{\theta}. \end{aligned} \tag{3.7.51}$$

b)

Now we have to derive J and ρ in both frames. Let's start from the beginning. In a rest frame (so we are co-moving with the wire), we know that there is an equation of state for the current that says:

$$\vec{\nabla}\vec{J} + \partial_t \rho = 0. \tag{3.7.52}$$

So, basic electromagnetism problem. If we know the electric density, we know the current. We know that the charge density is given by:

$$\rho = \tfrac{q}{L}, \qquad \text{but} \quad r'_{\text{charge}} = 0 \quad \rightarrow \quad \rho = \tfrac{q\delta(r'-0)}{L}. \tag{3.7.53}$$

Where we have accounted for the location of the charge along the wire in the z-axis. As we can see in eq (**??**), there is no time dependence, so $\vec{J} = 0$, as expected. Things completely change if we move to the laboratory frame. Now the wire is moving respect to the observer with velocity $\vec{v}$. It is better to use the four-vector formalism in this frame, as:

$$J^{\mu'} = (c\rho', J^{i'}) = (c\frac{q\delta(r'-0)}{L}, 0). \tag{3.7.54}$$

To obtain the laboratory frame data, we just have to boost our result in the direction z as:

$$\begin{aligned} J^{\nu} &= \Lambda^{\nu}_{\mu} J^{\mu'}, \\ J^{0} &= \Lambda^{0}_{0} J^{0} = \gamma J'^{0}, \\ J^{i} &= \Lambda^{3}_{\mu} J^{\mu'} = -\gamma\beta J^{0'}. \end{aligned} \tag{3.7.55}$$

Where we have to notice that $r' = r$ in both frames, as it direction is perpendicular to that of motion of the system. The four-current ends up to be:

$$J^{\mu} = \left(\frac{c\gamma q\delta(r-0)}{L}, 0, 0, -\frac{\gamma v_z q\delta(r-0)}{L}\right). \tag{3.7.56}$$

c)

This section is basic electromagnetism again. We just have to obtain $\vec{E}$ and $\vec{B}$ in the laboratory frame from the charge and current densities in the previous section.

For $\vec{E}$ we just have to observe that $J^{0'} = \gamma J^{0}$. The only difference is how the charge densities relate between frames. So we can conclude that

$$\vec{E} = \gamma \vec{E}' = \frac{\gamma q}{2\pi r L}\hat{r}. \tag{3.7.57}$$

In the case of $\vec{B}$ we have to use Ampere's law. Recall that we are going to relate the magnetic field around a cycle to the current crossing a specific surface as:

$$\oint \vec{B}\cdot \vec{d}l = \mu_0 \int \vec{J}\cdot \vec{d}A = \mu_0 I. \tag{3.7.58}$$

So this is just (do not forget the jacobian when changing from cartesian delta to cylindrical one):

$$
\begin{aligned}
\vec{B}\cdot 2\pi r &= -\mu_0 \int \gamma \tfrac{v_z q}{L}\delta(r-0)\delta(\theta-0)\tfrac{1}{r}\cdot r dr\, d\theta = \\
&= -2\pi\,\mu_0\, 2\pi \underbrace{\int_0^\infty}_{sym} \tfrac{\gamma v_z q\delta(r-0)}{L} dr = \\
&= -\tfrac{4\pi^2\mu_0}{2}\int_{-\infty}^{\infty} \tfrac{\gamma v_z q\delta(r-0)}{L} dr = \\
\vec{B} &= -\tfrac{\mu_0\gamma v_z q}{\pi r L}\hat{\theta}.
\end{aligned}
\tag{3.7.59}
$$

(Missing 2 and c are due to different unit system for different computations.)

3.7.7 Relativistic Ohm's law

a)

We are asked to generalised the expression of the current for a wire from a more general reference system. A general expression for the current in any frame, given by the 4-vector notation is:

$$J^\mu = (\rho, J^i), \quad \text{With } J^i \text{ the usual three-dimensional current } \vec{J}. \tag{3.7.60}$$

We can see in the description of the statement that it contains a four-velocity term. Generally, this is of the form:

$$U^\mu = (1, \partial_t x^i). \tag{3.7.61}$$

In our case, this means that $U^{\mu'} = (c, 0)$, as the description is given in the rest frame. One question that we may have is: How does the electric field looks in this frame? Recall from previous exercises that $F^{0i} = -E^i$. At this stage we have some tools to "craft" the general covariant expression for J^μ.

Let's start by looking at something of the form:

$$-F^{'0i}U_{'0} = \eta_{00}F^{'0i}\eta_{00}U^{'0}. \tag{3.7.62}$$

Where the tilde stands for the rest frame. As we are in the rest frame, and U_0 is the only non-zero component of U^μ, we can generalise our expression to $F^{'\mu\nu}U_{'\nu}$. We can see that this does not affect, as:

$$F^{'\mu\nu}U_{'\nu} = F^{'\mu 0}U_{'0} + \cancel{F^{'\mu 1}U_{'1}} + \cdots = (0, E^{'i}). \tag{3.7.63}$$

And then, sure of this property, we can multiply by σ to get RHS of the desired expression.[30] This means we have something like this:

$$J^{\mu} = (?, J^{i}) = \sigma F^{'\mu\nu}U^{'}_{\nu}. \tag{3.7.64}$$

In a general frame, we will have the presence of the density around, but we want to get rid of it in the LHS of the expression. In order to do so, we can again exploit the fact that $U^{'i} = 0$. In this case, we can craft something like:

$$\begin{aligned} J^{'\mu} &= (\rho', J^{'i}) \quad \text{Contract both sides by } U^{'}_{\mu}, \\ J^{'\mu}U^{'}_{\mu} &= (\rho^{i}, J^{'i}) \underbrace{(-1, 0)}_{U_0 = \eta_{00}U^0} = -\rho' \end{aligned} \tag{3.7.65}$$

As we do not want ρ to appear at LHS of the covariant expression, we can add this term to RHS. This terms should be multplied by a velocity, as the density will be moving depending to the reference system used. We have then:

$$\begin{aligned} (0, J^{i}) &= J^{'\nu} - (J^{'\mu}U^{'}_{\mu})U^{'\nu}, \\ \sigma F^{'\mu\nu}U^{'}_{\nu} &= J^{'\nu} - (J^{'\mu}U^{'}_{\mu})U^{'\nu}. \end{aligned} \tag{3.7.66}$$

b)

So far we have just expressed $\vec{J}$ in the rest frame in a fancy four-dimensional way. But, what happens to the expression if we boost to a laboratory frame and we say that the material moves with $\vec{v} = c\vec{\beta}$? We know so far that in the laboratory frame:

$$U^{\mu} = (\gamma, \gamma\vec{v}), \qquad J^{\mu} = (c\rho, \vec{J}). \tag{3.7.67}$$

We can take our previous results and try to impose these general expressions. Although we are asked to find the expression for $\vec{J}$, let's calculate also J^0 (SPOILER: It will help us to simplify $\vec{J}$).

[30] Observe that we work now with cgs system, so no c around.

$$
\begin{aligned}
J^0 - \tfrac{1}{c^2}(U_\nu J^\nu)U^0 &= \tfrac{\sigma}{c} F^{0\nu} U_\nu, \\
J^0 - \tfrac{1}{c^2}(U_0 J^0 + U_i J^i)U^0 &= \tfrac{\sigma}{c}(F^{0i} U_i + \cancel{F^{00} U_0}), \\
c\rho - \tfrac{1}{c^2}(-\gamma^2 c^4 \rho + \gamma^2 \vec{v}\vec{J} c) &= \tfrac{\sigma}{c}(-E^i)\gamma \vec{v}, \\
c^2 \rho \underbrace{(1+\gamma^2)}_{=-\beta^2\gamma^2} - \gamma^2 \vec{v}\vec{J} &= -\sigma \vec{E}\gamma\vec{v}, \\
-\vec{v}^2 \rho \gamma^2 - \gamma^2 \vec{v}\vec{J} &= -\sigma\vec{E}\gamma\vec{v}.
\end{aligned} \tag{3.7.68}
$$

For the case of J^μ we have:

$$
\begin{aligned}
J^i - \tfrac{1}{c^2}(U_\nu J^\nu)U^i &= \tfrac{\sigma}{c} F^{i/n} U_\nu, \\
J^i - \tfrac{1}{c^2}(U_0 J^0 + U_i J^i)U^i &= \tfrac{\sigma}{c}(F^{i0} U_0 + F^{ij} U_j), \\
J^i - \tfrac{1}{c^2}(-c^2 \rho\gamma + J^i \vec{v})\gamma\vec{v} &= \tfrac{\sigma}{c}(E^i \gamma c + \underbrace{\epsilon^{ijk} B_k \gamma v_j}_{\vec{v}\times\vec{B}}), \\
\vec{J} + \rho\gamma^2 \vec{v} - \gamma^2 \tfrac{1}{c^2}\vec{v}(\vec{v}\cdot\vec{J}) &= \sigma\gamma(\vec{E} + \tfrac{1}{c}\vec{v}\times\vec{B}).
\end{aligned} \tag{3.7.69}
$$

We can further simplify eq (**??**) by taking $\vec{v}\cdot\vec{J}$ in eq (**??**) and substitute. Then one has to massage the resulting RHS to obtain:

$$
\vec{J} = \sigma\gamma(\vec{E} - \gamma\beta^2\vec{E} + \vec{\beta}\times\vec{B}) - \vec{v}\rho. \tag{3.7.70}
$$

As we wanted to find. Observe that $\vec{J}$ has changed. Now we have the electric field contracted due to the Lorentz factor. $\vec{B}$ also joined the party.

c)

Let's assume now that the medium is uncharged in the rest frame, so $\rho' = 0$. How does ρ and $\vec{J}$ look like from the laboratory frame? As $\rho' = 0$ this means that:

$$
J^{\mu'} = (0, \vec{J}'), \qquad U^{\mu'} = (c, 0) \quad \rightarrow \quad J^{\mu'} U_{\mu'} = 0. \tag{3.7.71}
$$

In the laboratory frame, things look like:

$$
J^\mu = (c\rho, \vec{J}), \qquad U^\mu = (c\gamma, \vec{v}\gamma). \tag{3.7.72}
$$

So, we can see that ρ and $\vec{J}$ are of the form:

$$\begin{aligned} J^0 &= \rho = \sigma F^{\nu 0} U_\nu = \sigma\gamma\vec{v}\cdot\vec{E}, \\ J^i &= \vec{J} = \sigma F^{\nu i} U_\nu = \sigma\gamma(\vec{E}+\vec{v}\times\vec{B}). \end{aligned} \tag{3.7.73}$$

3.7.8 E: A Loooooong Cylinder and Several Frames

UNDER CONSTRUCTION

3.7.9 E: Planes and Frames

UNDER CONSTRUCTION

3.7.10 E: Different Points of View

1):

This first part of the problem could be said to be straightforward. One just have to boost J^μ to a new frame with Λ in the z-direction, so:

$$J^\nu_{boost} = \Lambda^\nu_\mu J^\mu = \left(\gamma\left(c\lambda - \tfrac{v}{c}I\right), 0, 0, \gamma\left(I - v\lambda\right)\right). \tag{3.7.74}$$

2):

In this part of the problem one can be confused and may try to compute things when they are not needed. If we want to boost to a frame where $\mathbf{E}'$ vanishes, this means that the charge linear density λ' in that frame should be equal to "0". By just looking at previous section results, this requires $I > c\lambda$. On the other hand, if we want to find $\mathbf{B}'$ boosting to a specific frame, we just have to recall Biot-Savart and check that $I' = 0$ in this frame. This will happen when $c\lambda > I$.These are the conditions we need to impose.

3.7.11 E: Waves Across Reference Frames

UNDER CONSTRUCTION

3.8 Lagrangian Manipulations

3.8.1 A Relativistic Particle Coupled to a Scalar Field

We have the following:

$$\mathscr{S} = -mc^2 \int ds - g \int ds \varphi(\mathbf{r}(s)). \tag{3.8.1}$$

The first term is the action of a free point particle. The Lagrangian of the latter is $L_p = -mc^2/\gamma$. Therefore, in terms of the proper time differential $d\tau = dt/\gamma$,

$$\mathscr{S}_p = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int d\tau = -mc \int d(c\tau). \tag{3.8.2}$$

We conclude that $ds = cd\tau$ and the total Lagrangian can be written as a function of time:

$$L = -\frac{mc^2}{\gamma} - \frac{gc}{\gamma}\varphi(\mathbf{r}(t)), \tag{3.8.3}$$

Where $\mathbf{r}$ is the canonical coordinate and $\mathbf{v}$ (inside γ) is the canonical momenta. So now, we just have to calculate the Euler-Lagrange equation to get the EOM.

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} &= 0, \\
\frac{d}{dt}\left(\gamma m\mathbf{v} + \gamma g\frac{\mathbf{v}}{c}\varphi\right) - \frac{gc}{\gamma}\vec{\nabla}\varphi &= 0, \\
\frac{d}{dt}(\gamma m\mathbf{v}c) = -\left(\frac{d}{dt}\left(g\gamma\frac{\mathbf{v}}{c}\varphi\right) + \frac{gc}{\gamma}\nabla\varphi\right)&.
\end{aligned} \tag{3.8.4}$$

Observe that the first term corresponds to the Coulomb "Force", while the last one to the right seems like the $\mathbf{E}$ -field with a φ as potential. One can think of the remaining term as a correction to the Coulomb force expression.

3.8.2 One-Dimensional Massive Scalar Field

This is a basic problem to get used to Lagrangian (and in this case Hamiltonian) manipulation. If we isolate the Lagrangian density, we have:

$$\mathscr{L} = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - m^2\varphi^2\right]. \tag{3.8.5}$$

As the only canonical coordinate is $\varphi(x, t)$, we just have one EOM given by:

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\varphi}} - \frac{\partial \mathscr{L}}{\partial \varphi} + \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial(\partial\varphi/\partial x)} = \\
&= \frac{1}{c^2}\frac{d}{dt}\dot{\varphi} + m^2\varphi - \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial x} = \\
&= \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + m^2\varphi = \\
\text{EOM} &= \frac{1}{c^2}\ddot{\varphi} - \frac{\partial^2\varphi}{\partial x^2} + m^2\varphi = 0.
\end{aligned}
\tag{3.8.6}
$$

Now we generalise the momentum using $\pi = \partial\mathscr{L}/\partial\dot{\varphi} = \dot{\varphi}/c^2$. Therefore, the Hamiltonian density is:

$$
\begin{aligned}
\mathscr{H} &= \pi\dot{\varphi} - \mathscr{L} = \\
&= \pi\frac{\partial\varphi}{\partial t} - \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2 - m^2\varphi^2\right] = \\
&= \frac{1}{2}\left[c^2\pi^2 + (\partial\varphi/\partial x)^2 + m^2\varphi^2\right].
\end{aligned}
\tag{3.8.7}
$$

And the Hamilton equations are given by:

$$
\dot{\pi} = -\frac{\partial\mathscr{H}}{\partial\varphi} + \frac{\partial}{\partial x}\frac{\partial\mathscr{H}}{\partial(\partial\varphi/\partial x)} = -m^2\varphi + \frac{\partial^2\varphi}{\partial x^2}. \tag{3.8.8}
$$

$$
\dot{\varphi} = \frac{\partial\mathscr{H}}{\partial\pi} - \frac{\partial}{\partial x}\frac{\partial\mathscr{H}}{\partial(\partial\pi/\partial x)} = c^2\pi. \tag{3.8.9}
$$

We can go further and derive (**??**) respect to t to arrive to the final expression

$$
\frac{1}{c^2}\ddot{\varphi} + m^2\varphi - \frac{\partial^2\varphi}{\partial x^2} = 0. \tag{3.8.10}
$$

Which exactly correponds to the EOM derived by the Lagrangian method.

3.8.3 Introduction to Lagrangian Manipulations

a): In this first part of this problem, we will learn how to deal with indices manipulation in a deeper way than previous exercises. But first, the important things. Recall that Equations Of Motion (EOM) are given by the Euler-Lagrange equation as:

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathscr{L}}{\partial \phi} = 0. \tag{3.8.11}$$

Where ϕ corresponds to the canonical coordinates in our theory. In this case corresponds to the 4-vector field A_μ. Then, we just have to move the wheel and produce some terms.

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_{å} A_\beta \partial^{å} A^\beta) = \\
&= \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_{å} A_\beta \eta^{å\gamma}\eta^{\beta\zeta}\partial^\gamma A^\zeta) = \\
&= \partial_\mu(\delta^\mu_{å}\delta^\nu_\beta \eta^{å\gamma}\eta^{\beta\zeta}\partial_\gamma A_\zeta + \delta^\mu_\gamma \delta^\nu_\zeta \eta^{å\gamma}\eta^{\beta\zeta}\partial_{å} A_\beta) = \\
&= \text{Substitute indices in } \eta \text{ by those ones in the } \delta = \\
&= 2\partial_\mu(\partial^\mu A^\nu).
\end{aligned} \tag{3.8.12}$$

As for the other part of the EOM, we find:

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial A_n} &= J_{å}\frac{\partial}{\partial A_\nu} A^{å} = J_{å}\eta^{å\beta}\partial_{A_\nu} A_\beta = \\
&= J_{å}\eta^{å\beta}\delta^\nu_\beta = J_\nu.
\end{aligned} \tag{3.8.13}$$

So we find that the equation of motion for the field A_ν is given by (Factors have been removed from previous calculations. One has to introduce them back):

$$\frac{1}{4\pi}\partial_\mu \partial^\mu A^\nu = \frac{1}{c} J^\nu. \tag{3.8.14}$$

So, Does this look like Maxwell equations? Recall how they look like:

$$\begin{aligned}
\partial_\mu F^{\mu\nu} &= \mu_0 J^\nu, \\
\partial_\mu \star F^{\mu\nu} &= 0.
\end{aligned} \tag{3.8.15}$$

And what do we have (factors of π and c aside)? Let's carefully expand the first expression in (**??**). This is:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \underbrace{\partial_\mu \partial^\nu A^\mu}_{\text{What we do not have in ??}} = \mu_0 J^\nu. \tag{3.8.16}$$

So we can conclude that $\partial_\mu \partial^\nu A^\mu$ is 0 in our case. This corresponds to the so called Lorenz gauge ($\partial_\mu A^\mu = 0$). It is under this circumstance that the EOM **??** corresponds to Maxwell's equation.

b):

In order to show that both $\mathscr{L}$ differ by a four-divergence term (a.k.a $\partial_\mu v^\mu$), let us massage the well know $\mathscr{L}_{elec}$:

$$\mathscr{L}_{elec} = -\tfrac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \tfrac{1}{c} J_\alpha A^\alpha. \tag{3.8.17}$$

So our intuition should be pointing towards the first term in the previous expression. Can we transform this in such a way that it directly looks as a four-divergence is missing?

$$\begin{aligned}
-\tfrac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} &= -\tfrac{1}{16\pi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = \\
&= -\tfrac{1}{16\pi}(\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha - \partial_\beta A_\alpha \partial^\alpha A^\beta + \partial^\beta A^\alpha \partial_\beta A_\alpha) = \\
&= -\tfrac{1}{8\pi}(\partial_\alpha A_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha)).
\end{aligned} \tag{3.8.18}$$

So, if we take away both Lagrangians to see what the difference is, we will find that:

$$\Delta\mathscr{L} = \tfrac{1}{8\pi} \partial_\alpha A_\beta \partial^\beta A^\alpha. \tag{3.8.19}$$

We are getting closer. The idea would be now to rearrange the previous expression such that we can get something as $\partial_\mu A^\mu$, so we can claim for a missing four-divergence. At this stage we need some inspiration. Let assume derivative variations of the term $\underbrace{\partial_\alpha A_\beta \partial^\beta A^\alpha}_{\star}$ as:

$$\begin{aligned}
\partial_\alpha (A_\beta \partial^\beta A^\alpha) &= \underbrace{\partial_\alpha A_\beta \partial^\beta A^\alpha}_{\star} + \underbrace{A_\beta \partial_\alpha \partial^\beta A^\alpha}_{\star\star}, \\
\partial^\beta (A_\beta \partial_\alpha A^\alpha) &= \partial^\beta A_\beta \partial_\alpha A^\alpha + \underbrace{A_\beta \partial^\beta (\partial_\alpha A^\alpha)}_{\star\star}.
\end{aligned} \tag{3.8.20}$$

So we can use previous expressions to massage $\Delta\mathscr{L}$ to obtain something like:

$$\begin{aligned}\Delta\mathscr{L} &\propto (\partial^\alpha(A_\beta\partial^\beta A_\alpha) - A_\beta\partial_\alpha\partial^\beta A^\alpha),\\ &\propto (\partial^\alpha(A_\beta\partial^\beta A_\alpha) - \partial^\beta(A_\beta\partial_\alpha A^\alpha) + (\partial^\mu A_\mu)^2).\end{aligned} \tag{3.8.21}$$

So we set the Lorenz gauge to 0 ($\partial^\beta A_\beta = 0$), while one expects $\Delta\mathscr{L} = 0$, it is found that:

$$\Delta\mathscr{L} \propto \partial^\alpha(A_\beta\partial^\beta A_\alpha - A_\alpha\partial^\beta A_\beta). \tag{3.8.22}$$

So we would not be able to find the initial difference given in **??** if we apply the Lorenz gauge. This affects the EOM. This added four-divergence gives a surface term, which will have no contribution, as A_μ is demanded to fall off rapidly enough when going to ∞[31].

So as the action of the divergence part goes to 0, it is not affected, neither the equations of motion.

3.8.4 Coupling Extra Fields to A_μ

a): For the action to be Lorentz invariant it is required to be a scalar. We can apply some intelligence and divide the action in each of its terms. If all terms behave as a scalar, the whole action will and it will be Lorentz invariant. Let's analyse term by term:

- $\partial_\mu a\partial^\mu a$: We know that $a(x)$ is a scalar field. We know that $\partial_\mu a(x)$ is a vector field... But it is contracted with $\partial^\mu a(x)$, so the result is a scalar.

- $F_{\mu\nu}F^{\mu\nu}$ is also a scalar, as its indices are contracted.

- $aF_{\mu\nu}\star F^{\mu\nu}$ it is not so straightforward to see. In this case, we have to know that the Hodge star $\star$ changes the sign under a coordinate reflection $\vec{x}\to-\vec{x}$. If this is the case, any form that transform in this way will be called pseudo-form. This pseudo-feature is inherited through products and combinations of forms. Hence $F_{\mu\nu}\star F^{\mu\nu}$ is a pseudo-scalar. One fancy thing here is that the combination of two pseudo-scalars gives a pure scalar. This can be used to require $a(x)$ to be a pseudo-scalar, to preserve Lorentz invariance in the whole action.

[31] Other option would be to demand compact support of that form in a given region or computing everything on a compact manifold, which is not the case for our regular space-time... Or is it?

- $\partial_\mu\left(aA_\nu\tilde{F}^{\mu\nu}\right)$ works as in the previous point. Pseudo-scalar × pseudo-scalar = scalar.

b):

First of all, we have to realise that the Lagrangian comes within the action. This one is given by:

$$\mathscr{L} = -\frac{1}{2}\partial_\mu a\partial^\mu a - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{f}\left[aF_{\mu\nu}\star F^{\mu\nu} - 2\partial_\mu\left(aA_\nu\star F^{\mu\nu}\right)\right]. \tag{3.8.23}$$

And the Euler-Lagrange equation giving EOM's is:

$$\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\right) - \frac{\partial\mathscr{L}}{\partial\phi} = 0. \tag{3.8.24}$$

In order to write down the EOM's, we have to realise that we have two canonical coordinates ϕ in this exercise; $a(x)$ and A_μ. So we have:

$$\begin{aligned}\partial_\mu\left(\tfrac{\partial\mathscr{L}}{\partial(\partial_\mu a)}\right) - \tfrac{\partial\mathscr{L}}{\partial a} = 0,\\ \partial_\mu\left(\tfrac{\partial\mathscr{L}}{\partial(\partial_\mu A_\nu)}\right) - \tfrac{\partial\mathscr{L}}{\partial A_\nu} = 0.\end{aligned} \tag{3.8.25}$$

Before we start computing like crazy, lets carefully observe the last term of the Lagrangian **??**. If we massage it...

$$\partial_\mu\left(aA_\nu\star F^{\mu\nu}\right) = \partial_\mu\left(aA_\nu\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\right) = \partial_\mu\left(a\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right). \tag{3.8.26}$$

This is a total derivative that will no contribute to the EOM's, as its initial and final terms are equivalent. So, for the equation of motion of the axion $a(x)$ we can compute it to be:

$$\begin{aligned}-\frac{1}{2}\partial_\mu\left[\frac{\partial\left(\partial_\tau a\right)}{\partial\left(\partial_\mu a\right)}\partial^\tau a + \partial_\tau a\frac{\partial\left(\partial^\tau a\right)}{\partial\left(\partial_\mu a\right)}\right] - \left(-\frac{1}{f}F_{\rho\sigma}\star F^{\rho\sigma}\right) = 0,\\ -\frac{1}{2}\partial_\mu\left(2\partial^\tau a\,\delta^\mu_\tau\right) + \frac{1}{f}F_{\rho\sigma}\star F^{\rho\sigma} = -\partial_\mu\partial^\mu a + \frac{1}{f}F_{\rho\sigma}\star F^{\rho\sigma} = 0,\\ \Box a = \frac{1}{f}F_{\rho\sigma}\star F^{\rho\sigma}.\end{aligned} \tag{3.8.27}$$

For the equation of motion of A_ν we have to observe that there is no dependence on A_ν in $\mathscr{L}$. Those are good news; We do not have to compute too much. This is:

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu a \right)} \right) = 0. \tag{3.8.28}$$

Then the equation of motion is:

$$\partial_\mu \left(-\frac{1}{4} \underbrace{\frac{\partial \left(F_{\rho\sigma} F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)}}_{\mathrm{I}} - \frac{a}{f} \underbrace{\frac{\partial \left(F_{\rho\sigma} \star F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)}}_{\mathrm{II}} \right) = 0. \tag{3.8.29}$$

It looks quite involved, so let's apply a little bit of *divide et vinces*.

- **I**:

$$\begin{aligned}
\frac{\partial \left(F_{\rho\sigma} F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} &= F_{\rho\sigma} \frac{\partial \left(F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} + \frac{\partial \left(F_{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} F^{\rho\sigma} = \\
&= 2 \frac{\partial \left(F_{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} F^{\rho\sigma} = 2 \frac{\partial \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho \right)}{\partial \left(\partial_\mu A_\nu \right)} F^{\rho\sigma} = 2 \frac{\partial \left(2 \partial_\rho A_\sigma \right)}{\partial \left(\partial_\mu A_\nu \right)} F^{\rho\sigma} = \\
&= 4 F^{\rho\sigma} \delta^\mu_\rho \delta^\nu_\sigma = 4 F^{\mu\nu}.
\end{aligned} \tag{3.8.30}$$

- **II**:

$$\begin{aligned}
\frac{\partial \left(F_{\rho\sigma} \star F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} &= F_{\rho\sigma} \frac{\partial \left(\star F^{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} + \frac{\partial \left(F_{\rho\sigma} \right)}{\partial \left(\partial_\mu A_\nu \right)} \star F^{\rho\sigma} = \\
&= F_{\rho\sigma} \frac{\partial \left(\frac{1}{2} \epsilon^{\rho\sigma\alpha\beta} F_{\alpha\beta} \right)}{\partial \left(\partial_\mu A_\nu \right)} + 2 \star F^{\rho\sigma} \delta^\mu_\rho \delta^\nu_\sigma = 2 \frac{1}{2} \epsilon^{\rho\sigma\alpha\beta} F_{\rho\sigma} \delta^\mu_\alpha \delta^\nu_\beta + 2 \star F^{\mu\nu} = \\
&= 4 \star F^{\mu\nu}.
\end{aligned} \tag{3.8.31}$$

Then, puting $I + II$ together one gets the final result:

$$
\begin{aligned}
\partial_\mu\left[-F^{\mu\nu}-\frac{4a}{f}\star F^{\mu\nu}\right]&=0,\\
-\partial_\mu F^{\mu\nu}-\frac{4}{f}\left(\partial_\mu a\right)\star F^{\mu\nu}-\frac{4a}{f}\cancel{\partial_\mu\star F^{\mu\nu}}&=0,\\
\partial_\mu F^{\mu\nu}&=-\frac{4}{f}\left(\partial_\mu a\right)\star F^{\mu\nu}.
\end{aligned}
\tag{3.8.32}
$$

c): To show that the action is invariant under a scalar transformation of the form $a\to a+\epsilon$, we just have to check that all terms in the action remain the same. This means:

- $$\partial_\mu(a+\epsilon)=\partial_\mu a+\cancel{\partial_\mu\epsilon}=\partial_\mu a. \tag{3.8.33}$$

- $$(a+\epsilon)F_{\mu\nu}\star F^{\mu\nu}=aF_{\mu\nu}\star F^{\mu\nu}+\epsilon F_{\mu\nu}\star F^{\mu\nu}. \tag{3.8.34}$$

- $$-2\partial_\mu\left[(a+\epsilon)A_\nu\star F^{\mu\nu}\right]=-2\partial_\mu\left[aA_\nu\star F^{\mu\nu}\right]-2\partial_\mu\left[\epsilon A_\nu\star F^{\mu\nu}\right]. \tag{3.8.35}$$

We can see that it looks like there are two terms that still contain an ϵ. Do they anhilite each other? Let's study it. We can call K to this term and expand.

$$
\begin{aligned}
K=\epsilon F_{\mu\nu}\star F^{\mu\nu}-2\partial_\mu\left[\epsilon A_\nu\star F^{\mu\nu}\right]&=\epsilon F_{\mu\nu}\star F^{\mu\nu}-2\cancel{\left(\partial_\mu\epsilon\right)}A_\nu\star F^{\mu\nu}\\
&\quad-2\left(\partial_\mu A_\nu\right)\epsilon\star F^{\mu\nu}-2\epsilon\cancel{\left(\partial_\mu\star F^{\mu\nu}\right)}A_\nu=\\
&=\epsilon F_{\mu\nu}\star F^{\mu\nu}-2\epsilon\underbrace{\partial_\mu A_\nu}_{=\frac{1}{2}F_{\mu\nu}}\star F^{\mu\nu}=0
\end{aligned}
\tag{3.8.36}
$$

Everything cancels in the end, so the action is invariant under this transformation. And we know that every invariance of the action has a conserved charged asociated to a Noether current.

d):

While this part of the exercise is beyond the scope of this course and problems, we can always solve it, for the pleasure of those with knowledge gluttony. We know that the current is:

$$j^{\mu} = \mathscr{L}\frac{\delta x^{\mu}}{\delta\epsilon} + \frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}\phi\right)}\frac{\delta_0\phi}{\delta\epsilon}. \tag{3.8.37}$$

Associated changes to the coordinates respect to the field displacement in the previous section are given by:

$$\frac{\delta x^{\mu}}{\delta\epsilon} = 0; \quad \frac{\delta_0 a}{\delta\epsilon} = 1; \quad \frac{\delta_0 A_{\nu}}{\delta\epsilon} = 0. \tag{3.8.38}$$

So the current ends up to be:

$$\begin{aligned} j^{\mu} &= \frac{\partial\mathscr{L}}{\partial\left(\partial_{\mu}a\right)}, \\ &= -\frac{1}{2}\partial^{\mu}a + \frac{2}{f}A_{\nu}\star F^{\mu\nu}. \end{aligned} \tag{3.8.39}$$

3.8.5 E: Ponderous Light

1):

To prove this invariance, we just have to go by pieces of the whole action. We know:

$$F_{\mu\nu} \to \partial_{\mu}A_{\nu} + \partial_{\mu}\partial_{\nu}\alpha - \partial_{\nu}A_{\mu} - \partial_{\nu}\partial_{\mu}\alpha = F_{\mu\nu} + (\partial_{\mu}\partial_{\nu}\alpha - \partial_{\nu}\partial_{\mu}\alpha) = F_{\mu\nu}. \tag{3.8.40}$$

Similarly, for the term with the current:

$$J^{\mu}A_{\mu} \to J^{\mu}A_{\mu} - J^{\mu}\partial_{\mu}\alpha = J^{\mu}A_{\mu} - \left(\underbrace{\partial_{\mu}(J^{\mu}\alpha)}_{\text{Stokes}=0} - \alpha\underbrace{\partial_{\mu}J^{\mu}}_{=0}\right) = J^{\mu}A_{\mu}. \tag{3.8.41}$$

And finally, checking the problematic term we find:

$$\partial_{\mu}\phi - mA_{\mu} \to \partial_{\mu}\phi - mA_{\mu} + m\left(\partial_{\mu}\alpha - \partial_{\mu}\alpha\right) = \partial_{\mu}\phi - mA_{\mu}. \tag{3.8.42}$$

So our action is invariant under such gauge transformation. How does it looks like if we fix $\phi = 0$? It looks like:

$$\mathcal{L} = \tfrac{m^2}{8\pi} A_\mu A^\mu - \tfrac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \tfrac{1}{c} J^\mu A_\mu. \tag{3.8.43}$$

2):

Again, here we do not have to work too much. The second and third term in the previous Lagrangian will give rise to the well know equation of motion of Electromagnetic theory. The first one is something new. This will contribute as:

$$\text{EOM} \quad \frac{m^2}{4\pi} A^\mu + \frac{1}{4\pi} \partial_\nu F^{\mu\nu} = \frac{J^\mu}{c}. \tag{3.8.44}$$

3):

Doing as the problem says, lets contract both sides with a partial derivative:

$$\frac{m^2}{4\pi} \partial_\mu A^\mu + \frac{1}{4\pi} \underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{=0 \text{ by antisym}} = \underbrace{\frac{\partial_\mu J^\mu}{c}}_{=0} \to \partial_\mu A^\mu = 0. \tag{3.8.45}$$

Inserting back this Lorentz gauge into the EOM we find:

$$\left(\Box + m^2\right) A^\mu = \frac{4\pi}{c} J^\mu. \tag{3.8.46}$$

4):

So we have to basically solve here the previous equation when RHS is equal to 0. Recall that a plane wave is of the form:

$$A^\mu = \mathscr{A}^\mu e^{i\left(\vec{k}\vec{x} - \omega t\right)}. \tag{3.8.47}$$

Which plugged into expression (**??**) yields:

$$\omega^2 = c^2 (k^2 + m^2). \tag{3.8.48}$$

For the polarisation it is a little bit trickier. In this case one has to look at the Lorentz gauge condition $\partial_\mu A^\mu = 0$ to find that:

$$\frac{-i}{c}\omega\mathscr{A}^0 + i\vec{k}\cdot\vec{A} = 0. \tag{3.8.49}$$

Solving this equation for $\mathscr{A}^0$ one finds 3 polarisations (one for each individual possible value of $\vec{k}$.)

3.9 Radiation and Relativistic Dynamics

3.9.1 Emission Rates by Lorentz Transformation

In the (momentary) rest frame of the electron, the particle acceleration is given by $F' = ma' = eE'$ and Larmor's formula gives the exact rate at which the particle radiates energy:

$$P' = \frac{dU'}{dt'} = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\left(\frac{dp'}{dt'}\right)^2 = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\frac{e^2E'^2}{m^2}. \tag{3.9.1}$$

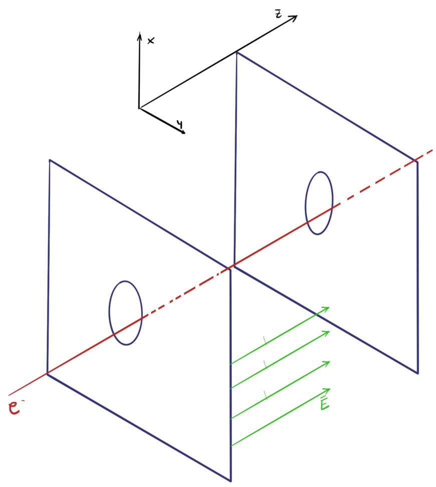


Figure 31: The system we want to study.

Therefore, the total energy lost to radiation (recall that $P = \frac{E}{t}$ and there is no change in $\vec{v}$ of the e^-) is;

$$\Delta U' = P't' = \frac{1}{4\pi\epsilon_0}\frac{2e^4E'^2t'}{3m^2c^3} \tag{3.9.2}$$

We have to realise that there is no preferred direction in the rest frame, so $\Delta P' = 0$. Transforming to the laboratory frame,

$$\Delta U = \gamma \Delta U' = \gamma \sqrt{\Delta U'^2 + \cancel{\Delta P'^2}} = \gamma \frac{1}{4\pi\epsilon_0} \frac{2e^4 E'^2 t'}{3m^2 c^3}. \tag{3.9.3}$$

The electric field $\vec{E}$ is parallel to the boost, so $E = E'$. The electron transit time through the capacitor is $t = d/v$ and $t = \gamma t'$ by time dilation. Therefore,

$$\Delta U = \gamma \frac{1}{4\pi\epsilon_0} \frac{2e^4 E'^2 t'}{3m^2 c^3} = \frac{1}{4\pi\epsilon_0} \frac{2e^4 E^2 d}{3m^2 c^3 v}. \tag{3.9.4}$$

The associated total momentum radiated is

$$\Delta P = \gamma \left(\cancel{\Delta P'} + v \Delta U'/c^2 \right) = \frac{\gamma v \Delta U'}{c^2} = \frac{v \Delta U}{c^2} = \frac{1}{4\pi\epsilon_0} \frac{2e^4 E^2 d}{3m^2 c^5}. \tag{3.9.5}$$

3.9.2 A Merry Go Round of Radiating Particles

For a single charged particle, the Liénard-Wiechert electric field is:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \left[\underbrace{\frac{(\hat{\mathbf{n}} - \boldsymbol{\beta})}{\gamma^2 (1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3 R^2}}_{\text{Static fields} \propto 1/R^2} + \underbrace{\frac{\hat{\mathbf{n}} \times \{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\}}{c(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3 R}}_{\text{Radiative fields} \propto 1/R} \right]_{\text{ret}} \tag{3.9.6}$$

Basically we have to prove that our final result only contains the first term. In order to do so, the first question we have to answer for this problem is: How does the set up look like? Observe the following sketch.

So we have to compute the position and velocity variables for a generic particle in this system. For simplicity, let us take the center of coordinates lying on the position $\vec{p}$ of the z axis. So $\mathbf{R}(t)$ is given by:

$$\mathbf{R}_k(t) = \vec{p} - \vec{q}_k(t) = -a \cos\left(\omega t + \phi_k\right) \hat{\mathbf{x}} - a \sin\left(\omega t + \phi_k\right) \hat{\mathbf{y}} + z \hat{\mathbf{z}}, \tag{3.9.7}$$

where $v = a\omega$ (ω is the angular velocity of these particles) and $\phi_k = 2\pi k/N$. From the sketch above, $R_k = \sqrt{a^2 + z^2} = R$ is the same for all the particles and constant, so $\hat{n} = \mathbf{R}/|\mathbf{R}|$. What is the velocity? And the acceleration?

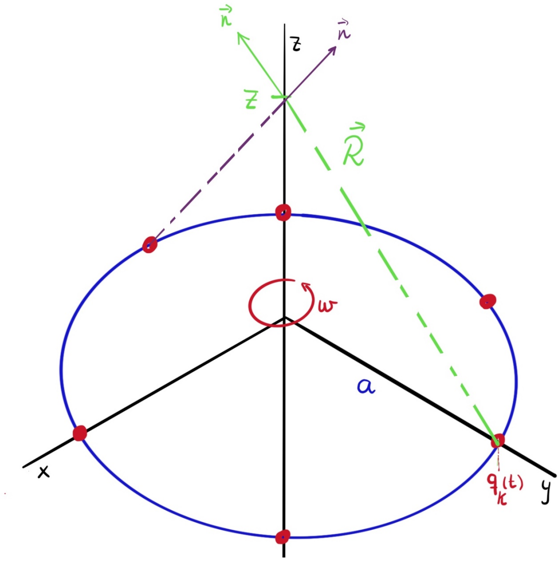


Figure 32: A frame of the motion of N particles around an axis.

$$\begin{aligned}\boldsymbol{\beta}_k &= -\frac{\partial \mathbf{q}_k}{c\,\partial t} = \beta\left[-\sin\left(\omega t+\phi_k\right)\hat{\mathbf{x}}+\cos\left(\omega t+\phi_k\right)\hat{\mathbf{y}}\right],\\ \dot{\boldsymbol{\beta}}_k &= -\frac{\partial \boldsymbol{\beta}_k}{\partial t} = -\omega\beta\left[\cos\left(\omega t+\phi_k\right)\hat{\mathbf{x}}+\sin\left(\omega t+\phi_k\right)\hat{\mathbf{y}}\right].\end{aligned} \tag{3.9.8}$$

Step by step, we get closer to the solution. Now, we have to compute the scalar and vector products as:

$$\begin{aligned}&\hat{\mathbf{n}}_k\cdot\boldsymbol{\beta}_k = (-\tfrac{a}{|\mathbf{R}|}\cos(\dots),-\tfrac{a}{|\mathbf{R}|}\sin(\dots),z)\cdot(-\beta\sin(\dots),\beta\cos(\dots),0)=0,\\ &\hat{\mathbf{n}}_k\times\left(\left(\hat{\mathbf{n}}_k-\boldsymbol{\beta}\right)\times\dot{\boldsymbol{\beta}}\right) = \underbrace{\left(\hat{\mathbf{n}}_k-\boldsymbol{\beta}\right)\left(\hat{\mathbf{n}}_k\cdot\dot{\boldsymbol{\beta}}\right)-\dot{\boldsymbol{\beta}}}_{\vec{a}\times\vec{b}\times\vec{c}=b(\vec{a}\cdot\vec{c})-c(\vec{a}\cdot\vec{b})}.\end{aligned} \tag{3.9.9}$$

Where:

$$\left(\hat{\mathbf{n}}_k\cdot\dot{\boldsymbol{\beta}}\right) = \frac{c\boldsymbol{\beta}^2}{|\mathbf{R}|}. \tag{3.9.10}$$

And $\left(\hat{\mathbf{n}}_k-\boldsymbol{\beta}\right)$ is left untouched for convenience. Then, after some computation we have all the necessary terms to compute the electric field, which is given by:

$$\mathbf{E}(z,t) = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{(\hat{\mathbf{n}}_k - \boldsymbol{\beta}_k)}{R^2}\left(\underbrace{\frac{1}{\gamma^2}+\beta^2}_{=1}\right) - \frac{\dot{\boldsymbol{\beta}}_k}{cR}\right]_{\text{ret}} = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{(\hat{\mathbf{n}}_k - \boldsymbol{\beta}_k)}{R^2} - \frac{\dot{\boldsymbol{\beta}}_k}{cR}\right]_{\text{ret}}. \tag{3.9.11}$$

At this point of the problem, we need to realise something. The x - and y -components of this electric field vanish because, when $N > 1$,

$$\sum_{k=1}^{N}\cos(\omega t + \phi_k) = \sum_{k=1}^{N}\sin(\omega t + \phi_k) = 0. \tag{3.9.12}$$

The proof that both sums vanish is something as:

$$\sum_{k=1}^{N} e^{i(\omega t+\phi_k)} = e^{i(\omega t + 2\pi/N)}\sum_{k=1}^{N} e^{i(k-1)2\pi/N} = e^{i(\omega t+2\pi/N)}\frac{1-e^{i2\pi}}{1-e^{i2\pi/N}} = 0. \tag{3.9.13}$$

One can also see this vanishing by considering an even[32] number of particles. In this case, by the radial symmetry in the system, each vector $\mathbf{R}_k$ will have a "counter" vector $\mathbf{R}_{-k}$, whose $\hat{x}$ and $\hat{y}$ components will have opposite sign of the k vector ones. So the only term in (**??**) which survives is the z -component of $\hat{\mathbf{n}}_k$. Hence, the electric field on the symmetry axis is:

$$\mathbf{E}(z,t) = \hat{z}\frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{z}{R^3}\right]_{\text{ret}} = \frac{qNz}{4\pi\epsilon_0 R^3}\hat{z} \quad (N > 1). \tag{3.9.14}$$

Which has not time component involved, neither explicit nor implicit. This indicates a static configuration of the electric field.

3.9.3 The Direction of the Velocity Field

The velocity field is given by:

$$\mathrm{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}-\boldsymbol{\beta}}{\gamma^2 g^3 R^2}\right]_{\text{ret}} \tag{3.9.15}$$

[32] In the case of odd number, one can take 3 particles and check that the sum of 2 of the i-components correspond to the remaining one with oposite sign.

And the direction of this field is the same as the direction of the vector,

$$[\mathbf{R}-\boldsymbol{\beta}R]_{\text{ret}} = \mathrm{r}-\mathrm{r}_0\,(t_{\text{ret}})-\boldsymbol{\beta}\,|\mathrm{r}-\mathrm{r}_0\,(t_{\text{ret}})|\,. \tag{3.9.16}$$

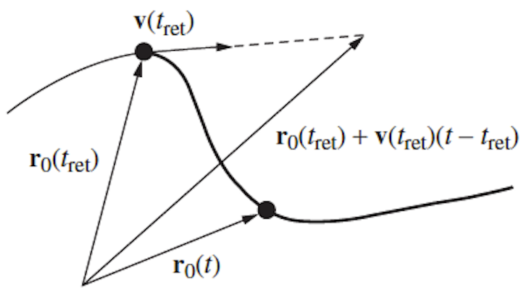
For our convenience, let the observer sit at the origin (r = 0). Then, because the retarded time is defined as:

$$t_{\text{ret}}+r_0\,(t_{\text{ret}})\,/\,c-t=0, \tag{3.9.17}$$

the velocity electric field can be expressed as:

$$\begin{aligned}\mathbf{E}\propto\ -\mathrm{r}_0\,(t_{\text{ret}}\,)-\frac{\mathbf{v}\,(t_{\text{ret}}\,)}{c}r_0\,(t_{\text{ret}}\,) &= -\mathbf{r}_0\,(t_{\text{ret}}\,)-\mathbf{v}\,(t_{\text{ret}}\,)\,(t-t_{\text{ret}}\,) = \\ &= -\left[\mathbf{r}_0\,(t_{\text{ret}}\,)+\mathbf{v}\,(t_{\text{ret}}\,)\,(t-t_{\text{ret}}\,)\right] = -\mathbf{r}_A.\end{aligned} \tag{3.9.18}$$

The diagram below shows that this proves the assertion.



3.9.4 Radiating 14.4 Jackson Problem

a):

In the non-relativistic limit, the radiated power is given by

$$\frac{dP(t)}{d\Omega} = \frac{e^2}{4\pi c}|\hat{n}\times\dot{\vec{\beta}}|^2. \tag{3.9.19}$$

For this part of the problem, we just have to make the observation that the particle has an harmonic motion along the z axis, so we take:

$$\vec{r} = \hat{z} a \cos\omega_0 t, \quad \vec{\beta} = \frac{\partial \vec{r}}{c\,\partial t} = -\hat{z}\frac{a\omega_0}{c}\sin\omega_0 t, \quad \dot{\vec{\beta}} = -\hat{z}\frac{a\omega_0^2}{c}\cos\omega_0 t. \tag{3.9.20}$$

By symmetry, we can assume the observer is in the $x-z$ plane tilted with angle θ from the vertical. This means that we take the normal $\hat{n}$ to be:

$$\hat{n} = \hat{x}\sin\theta + \hat{z}\cos\theta. \tag{3.9.21}$$

Now we have all the ingredients to cook up the formula (**??**), so:

$$\hat{n} \times \dot{\vec{\beta}} = \frac{a\omega_0^2}{c}\sin\theta\cos\omega_0 t\,\hat{y}, \quad \Rightarrow \quad \frac{dP(t)}{d\Omega} = \frac{e^2 a^2 \omega_0^4}{4\pi c^3}\sin^2\theta\cos^2\omega_0 t. \tag{3.9.22}$$

And then, taking a time average $\left(\cos^2\omega_0 t \to 1/2\right)$ this gives:

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \omega_0^4}{8\pi c^3}\sin^2\theta. \tag{3.9.23}$$

To get the total power radiated, we just have to integrate over the solid angle Ω, resulting:

$$P = \frac{e^2 a^2 \omega_0^4}{3c^3}. \tag{3.9.24}$$

We can now plot how the radiated energy per solid angle looks like,

b):

Now the particle decides to move along a circle of radius R with constant angular frecuency ω_0. This means that its position vector $\vec{r}$ is given by:

$$\vec{r} = R\left(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t\right). \tag{3.9.25}$$

Therefore, the velocity and the acceleration derived from this previous expression are:

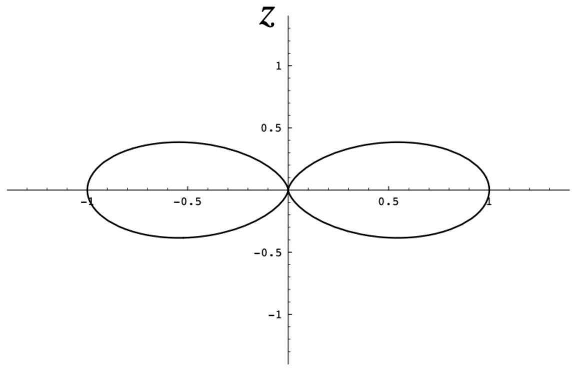


Figure 33: Distribution of the radiation for a bouncing particle.

$$\vec{\beta} = \frac{R\omega_0}{c}\left(-\hat{x}\sin\omega_0 t + \hat{y}\cos\omega_0 t\right) \quad \rightarrow \quad \dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}\left(\hat{x}\cos\omega_0 t + \hat{y}\sin\omega_0 t\right). \qquad (3.9.26)$$

Then, sitting at the same position as in the previous part of this exercise (a.k.a. same $\hat{n}$), one can find:

$$\hat{n}\times\dot{\vec{\beta}} = -\frac{R\omega_0^2}{c}\left[\cos\theta\cos\omega_0 t\,\hat{y} + (\sin\theta\,\hat{z} - \cos\theta\,\hat{x})\sin\omega_0 t\right]. \qquad (3.9.27)$$

And again, the distribution of radiation is:

$$\frac{dP(t)}{d\Omega} = \frac{e^2R^2\omega_0^4}{4\pi c^3}\left(\cos^2\theta\cos^2\omega_0 t + \sin^2\omega_0 t\right). \qquad (3.9.28)$$

Taking time average gives:

$$\frac{dP}{d\Omega} = \frac{e^2R^2\omega_0^4}{8\pi c^3}\left(1+\cos^2\theta\right). \qquad (3.9.29)$$

And the total radiation, after integrating is:

$$P = \frac{2e^2R^2\omega_0^4}{3c^3}. \qquad (3.9.30)$$

In this case, the distribution of radiation across the solid angle is a little bit more involved, but easy to obtain using Mathematica or something similar.

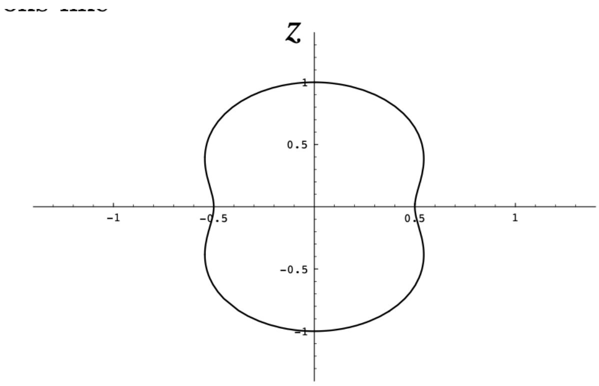


Figure 34: Distribution of the radiation for a particle in a circular motion.

3.9.5 A Fast Particle in a Constant Electric Field

The equation of motion is the basic one for a charged particle moving in a constant field **E**, as

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E}. \tag{3.9.31}$$

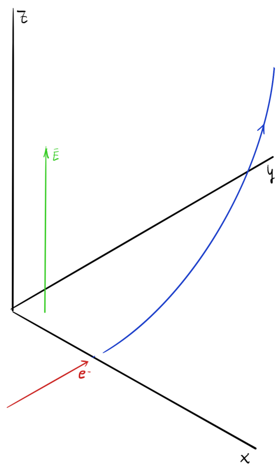


Figure 35: The motion of the particle within the field **E**.

With this, we can obtain the momentum of the particle depending on t. We just have to solve the differential equation imposing the initial condition $\mathbf{p}(0) = \gamma m u_0 \hat{y}$. Then we have:

$$\mathbf{p}(t) = p_0\hat{\mathbf{y}} + qEt\hat{\mathbf{z}}. \tag{3.9.32}$$

The initial (total) energy of the particle is $\mathscr{E}_0 = \sqrt{c^2 p_0^2 + m^2 c^4}$. Therefore, the instantaneous velocity of the particle is:

$$\mathbf{u}(t) = \frac{c^2\mathbf{p}}{\mathscr{E}} = \frac{c^2\mathbf{p}}{\sqrt{c^2p^2+m^2c^4}} = \frac{p_0\hat{\mathbf{y}} + qEt\hat{\mathbf{z}}}{\sqrt{\mathscr{E}_0^2 + c^2q^2E^2t^2}}c^2. \tag{3.9.33}$$

The particle speed goes $u \to c$ as $t \to \infty$. One can integrate the previous expression once respect to time to get $\mathbf{r}(t)$ as:

$$\mathbf{r}(t) = \hat{\mathbf{y}}\frac{cp_0}{qE}\sinh^{-1}\left(\frac{cqEt}{\mathscr{E}_0}\right) + \hat{\mathbf{z}}\frac{1}{qE}\sqrt{\mathscr{E}_0^2 + c^2q^2E^2t^2}. \tag{3.9.34}$$

The origin of coordinates so the integration constants are zero. Eliminating t and using the properties of $\sinh x$ and $\cosh y$ we can arrive to the expected expression if:

$$\begin{gathered}
(x,y,z) = \left(0, \frac{cp_0}{qE}\sinh^{-1}\left(\frac{cqEt}{\mathscr{E}_0}\right), \frac{1}{qE}\sqrt{\mathscr{E}_0^2 + c^2q^2E^2t^2}\right) \to \\
\left(\frac{qEy}{cp_0}\right) = \sinh^{-1}\left(\frac{cqEt}{\mathscr{E}_0}\right), \qquad z = \frac{1}{qE}\sqrt{\mathscr{E}_0^2 + c^2q^2E^2t^2}, \\
\cosh\left(\frac{qEy}{cp_0}\right) = \cosh\left(\sinh^{-1}\left(\frac{cqEt}{\mathscr{E}_0}\right)\right) = \sqrt{1 + \left(\frac{cqEt}{\mathscr{E}_0}\right)^2}, \\
\to z = \frac{\mathscr{E}_0}{qE}\cosh\left(\frac{qEy}{cp_0}\right).
\end{gathered} \tag{3.9.35}$$

The non-relativistic limit is $u \ll c$ or $cqEt \ll \mathscr{E}_0$. We recover the expected parabolic trajectory in this limit because $\cosh x \approx 1 + \frac{1}{2}x^2$ when $x \ll 1$.

3.9.6 A Ringy Radiating Problem

1):

In the rest frame, $\varphi' = 0$ and

$$\mathbf{A}'\left(\mathbf{r}'\right) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}' \times \mathbf{r}'}{r'^3}. \tag{3.9.36}$$

Therefore,

$$\mathbf{A}'_\perp = \frac{\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\parallel + \mathbf{m}'_\parallel \times \mathbf{r}'_\perp}{r'^3}, \qquad \mathbf{A}'_\parallel = \frac{\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\perp}{r'^3}. \tag{3.9.37}$$

On the other hand,

$$\begin{aligned} \mathbf{A}_\perp &= \mathbf{A}'_\perp, \\ \mathbf{A}_\parallel &= \gamma\left(\mathbf{A}'_\parallel + \mathbf{v}_0\varphi'/c^2\right) = \gamma\mathbf{A}'_\parallel, \\ \varphi &= \gamma\left(\varphi' + \mathbf{v}_0 \cdot \mathbf{A}'\right) = \gamma\mathbf{v}_0 \cdot \mathbf{A}'. \end{aligned} \tag{3.9.38}$$

All together becomes,

$$\begin{aligned} \mathbf{A}_\perp &= \frac{\mu_0}{4\pi}\frac{\mathbf{m}_\perp \times \gamma\left(\mathbf{r}_\parallel - \mathbf{v}_0 t\right) + \gamma\mathbf{m}_\parallel \times \mathbf{r}_\perp}{\left\{\gamma^2\left(\mathbf{r}_\parallel - \mathbf{v}_0 t\right)^2 + \mathbf{r}_\perp^2\right\}^{3/2}} = \frac{\gamma\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{R})_\perp}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}, \\ \mathbf{A}_\parallel &= \frac{\gamma\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\perp}{\left\{\gamma^2\left(\mathbf{r}_\parallel - \mathbf{v}_0 t\right)^2 + \mathbf{r}_\perp^2\right\}^{3/2}} = \frac{\gamma\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{R})_\parallel}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}. \end{aligned} \tag{3.9.39}$$

Adding up both components we get **A** as:

$$\mathbf{A} = \frac{\gamma\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{R}}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}. \tag{3.9.40}$$

Similarly,

$$\varphi = \mathbf{v}_0 \cdot \mathbf{A}_\parallel = \frac{\gamma\mu_0}{4\pi} \frac{v_0 \cdot (\mathbf{m} \times \mathbf{R})}{\left\{\gamma^2 \mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}. \tag{3.9.41}$$

2):

In the non-relativistic limit, $\gamma \to 1$, so

$$\begin{aligned} \mathbf{A} &= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{R}}{R^3}, \\ \varphi &= \frac{\mu_0}{4\pi} \frac{\mathbf{v}_0 \cdot (\mathbf{m} \times \mathbf{R})}{R^3} = \frac{1}{4\pi\epsilon_0} \frac{(\mathbf{v}_0 \times \mathbf{m}) \cdot \mathbf{R}/c^2}{R^3}. \end{aligned} \tag{3.9.42}$$

These are the vector and scalar potentials for a system moving at a velocity v_0 with a magnetic dipole moment $\mathbf{m}$ and an electric dipole moment $\mathbf{p} = (\mathbf{v}_0 \times \mathbf{m}) / c^2$.